Congruences on (n,m)-groups

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Abstract

In this paper we give a generalization of some definitions and properties of congruences on n-groups given in [2]. Congruences on (n,m)-groups are defined as congruences of the corresponding component algebra, and as kernel of a homomorphism, and a connection between these two definitions is given. Also, it is shown that for each congruence of an (n,m)-group $\mathbf{Q}$ there exists an invariant subgroup of its universal covering group $\mathbf{Q}^{\vee}$ of $\mathbf{Q}$, that is a subset of Q_{m+p}, where $m+p = sk$, $k = n-m > 0$. Conversely, for each invariant subgroup K of $Q^{\vee}$, which is a subset of Q_{m+p} and satisfies the condition

$$x_j y_j^{-1} \in K \,\&\, x_1 \cdots x_n = a_1 \cdots a_m, y_1 \cdots y_n = b_1 \cdots b_m \Rightarrow a_i b_i^{-1} \in K,$$

for all $i = \overline{1,m}$, there exists a congruence α on the (n,m)-group $\mathbf{Q}$, such that the corresponding invariant subgroup of $Q^{\vee}$ is exactly K.

1 Preliminary definitions and results

Let Q be a nonempty set. Denote by Q^r, r is a positive integer, the $r-$th Cartesian power of Q. Instead of denoting the elements of Q^r by $(a_1, \ldots, a_r)$ we will use the notation $a_1 \cdots a_r$, or a_1^r. In this way we can identify the set Q^r with the subset $\{a_1 \cdots a_r \mid a_\nu \varepsilon Q\}$ of the free semigroup Q^+ with a basis Q. The element $a_i \cdots a_j \in Q^+$ will be denoted by a_i^j, meaning the empty symbol when $j < i$, i.e. the unity of $Q^* = Q^+ \cup \{1\}, 1 \notin Q^+$.

Let $m, n, n-m = k > 0$ be positive integers, and $f : Q^n \to Q^m$ a mapping. Then we say that $\mathbf{Q} = (Q; f)$ is an (n,m)-groupoid, and f is an (n,m)-operation on Q. If, moreover, $\mathbf{Q}$ satisfies the condition

$$f(f(x_1^n)x_{n+1}^{2k+m}) = f(x_1^i f(x_{i+1}^{i+n})x_{i+n+1}^{2k+m}),$$

for each $1 \leq i \leq k$, $x_\nu \in Q$, then we say that f is an $associative(n,m)-operation$. We say that the ordered pair $(Q; f)$, where f[1] is an associative (n,m)-operation, is an (n,m)-*semigroup*.

[1]Further on we will denote an (n,m)-operation by [ ]

If $\mathbf{Q} = (Q; [\ ])$ is an (n,m)-semigroup, then the semigroup $\mathbf{Q}^\wedge$ given by the presentation

$$\mathbf{Q}^\wedge = \langle Q \mid \{(a_1^n, b_1^m); [a_1^n] = b_1^m\}\rangle$$

in the class of all semigroups, is said to be *the universal covering semigroup* of $\mathbf{Q}$.

The carrier $Q^\wedge$ of $\mathbf{Q}^\wedge$ is a disjoint union of the form

$$Q^\wedge = Q \cup Q^2 \cup \cdots \cup Q^m \cup Q_{m+1} \cup \cdots \cup Q_{m+k-1},$$

where $Q_{m+i} = Q^{m+i}/\beta$ such that β is the congruence on $\mathbf{Q}^\wedge$ induced by the defining relations of its presentation ([1]). We denote by $Q^\vee$ the subset $Q^m \cup Q_{m+1} \cup \cdots \cup Q_{m+k-1}$ of $Q^\wedge$. $Q^\vee$ is an ideal of $\mathbf{Q}^\wedge$ ([1]).

In this case we have the following property

$$i \leq m, x_\nu \in Q \Rightarrow (x_1 \cdots x_i = y_1 \cdots y_i \Rightarrow x_\nu = y_\nu), \nu = 1, i.$$

For each $(n,m)-$groupoid we can associate an algebra with m $n-$ary operations defined by

$$[a_1^n]_i = b_i \text{ iff } [a_1^n] = b_1^m,$$

where $i = \overline{1,m}$, and $a_\nu, b_\lambda \in Q$ [2]. then $(Q; [\ ]_1, \ldots, [\ ]_m)$ is called a *component algebra* of $\mathbf{Q}$.

Let $\mathbf{Q}$ and $\mathbf{Q}'$ be two $(n,m)-$semigroups and $\varphi : Q \to Q'$ a mapping. We say that φ is an *(n,m)- homomorphism* if it is a homomorphism between their corresponding component algebras. We can define a mapping $\varphi^\wedge$ between the corresponding universal covering semigroups by

$$\varphi^\wedge(x_1^i) = \varphi(x_1) \cdots \varphi(x_i). \tag{1.1}$$

Then $\varphi^\wedge$ is a *homomorphism induced by* φ.

We give, bellow, some connections between a homomorphism of (n,m)-semigroups and the induced one of their universal covering semigroups.

Prop 1.1 *Let $\varphi : Q \to Q'$ be a homomorphism of $(n,m)-$semigroups and $\varphi^\wedge$ the induced homomorphism defined by (1.1). Then $\varphi^\wedge$ is the unique homomorphique extension of φ and φ is surjective iff $\varphi^\wedge$ is surjective as well* ([1]).

Let $\mathbf{Q}$=$(Q; [\ ])$ be an $(n,m)-$semigroup, such that for all $a_\nu, b_\lambda \in Q$, there exist $x_i, b_i \in Q$, such that

$$[a_1^k x_1^m] = b_1^m = [y_1^m a_1^k].$$

Then we say that $\mathbf{Q}$ is an $(n,m) - group$.

In this case, when $\mathbf{Q}$ is an $(n,m)-$group, $\mathbf{Q}^\wedge$ is the universal covering semigroup and

$$\mathbf{Q}^\vee = Q^m \cup Q_{m+1} \cup \cdots \cup Q_{m+k-1}$$

[2]Further on we will assume that $a_\nu, b_\lambda \in Q$.

is a group, called *universal covering group of* $\mathbf{Q}$ ([1]).

For each $a \in Q$, $\mathbf{Q}^\vee$ has the form

$$Q^\vee = Q_m \cup aQ_m \cup \cdots \cup a_{k-1}Q_m,$$

and the unity $\mathbf{1}$ of $\mathbf{Q}^\vee$ is an element of Q_{m+p}, where $m+p < m+k$, and $m+p = sk$. Thus, Q can be considered as a subset of Q_{m+p+1}.

Prop 1.2 *Let* $\mathbf{Q} = (Q; [\,])$ *and* $\mathbf{Q}' = (Q'; [\,]')$ *be* $(n,m)-$*groups, and* $\varphi : Q \to Q'$ *a homomorphism. Then there exists a unique extension* $\varphi^\vee : Q^\vee \to Q'^\vee$ *of* φ, *defined by*

$$\varphi^\vee(x_1 \cdots x_{m+i}) = \varphi(x_1) \cdots \varphi(x_{m+i}),$$

and φ *is surjective (injective) iff* $\varphi^\vee$ *is surjective (injective) as well* ([1]).

2 Congruences on (n,m)-semigroups

Let α be an equivalence relation on Q. We say that α is a *congruence* on $\mathbf{Q}$ if for $i = \overline{1,n}$, we have

$$a_i \alpha b_i \Rightarrow [a_1^n]_j \alpha [b_1^n]_j, j = \overline{1,m},$$

i.e. if α is a congruence on the corresponding component algebra of $\mathbf{Q}$.

Let α be a congruence on an $(n,m)-$semigroup $\mathbf{Q}$, $\varphi = nat\alpha : Q \to Q/\alpha$ the natural homomorphism. Then $\varphi^\wedge : Q^\wedge \to (Q/\alpha)^\wedge$ is an epimorphism, and $\alpha^\wedge = ker\varphi^\wedge$ a congruence on $\mathbf{Q}^\wedge$. Thus

Prop 2.1 $(Q/\alpha)^\wedge \cong Q^\wedge/\alpha^\wedge$.

Using the properties of the universal covering semigroup of an *(n,m)*-semigroup and the definition of congruences of *(n,m)*- semigroups, we will give next some connections between congruences of the given *(n,m)*-semigroup and its universal covering semigroup.

Prop 2.2 *Let* β *be a congruence on* $\mathbf{Q}^\wedge$ *with the property*

$$x_j \beta y_j, j = \overline{1,n} \& x_1^n = a_1^m, y_1^n = b_1^m \Rightarrow a_i \beta b_i, i = \overline{1,m}. \tag{2.1}$$

Then $\alpha = \beta/Q$ *is a congruence on* $\mathbf{Q}$, *such that* $\alpha^\wedge \subseteq \beta$.

Prop 2.3 *Let* β *be a congruence of* $\mathbf{Q}^\wedge$, *such that satisfies (2.1) and*

$$x_\nu, y_\nu \in Q, i, j < m+k \Rightarrow (x_1^i \beta y_1^j \Rightarrow i = j). \tag{2.2}$$

Then $\alpha = \beta/Q$ *is a congruence on* $\mathbf{Q}$, *such that* $\alpha^\wedge = \beta$.

3 Congruences on (n,m)-groups

Let $\mathbf{Q}$ be an $(n,m)-$group, and α a congruence on $\mathbf{Q}$. Define a relation $\alpha^\vee$ on $\mathbf{Q}^\vee$ by

$$a^i x_1^m \alpha^\vee a^i y_1^m \Leftrightarrow x_i \alpha y_i, i = \overline{1,m}. \tag{3.1}$$

Then

Prop 3.1 *(i) $\alpha^\vee$ does not depend on the choice of a;*
(ii) $\alpha^\vee$ is a congruence on $\mathbf{Q}^\vee$, such that $\alpha^\vee/Q = \alpha$.

Thus, for each congruence α on an *(n,m)*-group, there is a congruence, namely $\alpha^\vee$, of its universal covering group, such that $\alpha^\vee/Q = \alpha$.

To be able to establish connections between the congruences of an *(n,m)*-group and its universal covering group, let us first give some properties of the congruence $\alpha^\vee$ on $\mathbf{Q}^\vee$ induced by a given congruence α of the given *(n,m)*-group $\mathbf{Q}$.

Prop 3.2 *Let α be a congruence on the* (n,m)*-group $\mathbf{Q}$, $\varphi = nat\alpha$, $\varphi^\vee : Q^\vee \to (Q/\alpha)^\vee$, and $\overline{\alpha} = ker\varphi^\vee$. Then $\overline{\alpha} = \alpha^\vee$.*

Prop 3.3 *Let α be a congruence on the* (n,m)*-group $\mathbf{Q}$. Then $\alpha^\vee$ satisfies the following conditions*

$$(i)\ x_j \alpha^\vee y_j \,\&\, x_1^n = a_1^m,\ y_1^n = b_1^m \Rightarrow a_i \alpha^\vee b_i,\ i = \overline{1,m}; \tag{3.2}$$

(ii) The invariant subgroup K, induced by $\alpha^\vee$ is a subset of Q_{m+p}, where $m + p = sk$, $0 \le p \le k-1$.

Prop 3.4 *Let β be a congruence on$\mathbf{Q}^\vee$, $\alpha = \beta/Q$ and β satisfies (3.2). Then α is a congruence on $\mathbf{Q}$, such that $\alpha^\vee \subseteq \beta$.*

The next proposition establishes connections under which the restriction $\alpha = \beta/Q$ of the congruence β of the universal covering group is such that $\alpha^\vee = \beta$.

Prop 3.5 *Let β be a congruence on $\mathbf{Q}^\vee$, $\alpha = \beta/Q$, β satisfies (3.2) and*

$$0 \le i,j < k \,\&\, x_1^{m+i}\, \beta\, y_1^{m+j} \Rightarrow i = j.$$

Then α is a congruence on $\mathbf{Q}$, such that $\alpha^\vee = \beta$.

Prop 3.6 *For each congruence α on the* (n,m)*-group $\mathbf{Q}$, there exists an invariant subgroup $K \subseteq Q_{m+p}$, such that*

$$x_j y_j^{-1} \in K \,\&\, x_1^n = a_1^m, y_1^n = b_1^m \Rightarrow a_i b_i^{-1} \in K,\ i = \overline{1,m}. \tag{3.3}$$

Coversely, for each invariant subgroup K of $\mathbf{Q}^\vee$, such tha $K \subseteq Q_{m+p}$ and (3.3) is satisfied, there exists a congruence α on the (n,m)*-group $\mathbf{Q}$, such that the invariant subgroup induced by $\alpha^\vee$ is exactly K.*

Thus Prop 3.6 is a characterization of the congruences of an *(n,m)*-group through invariant subgroups of its universal covering group.

As a corollary of Prop3.6 we obtain the following

Prop 3.7 *The lattice of congruences of an (n,m)-group is a modular one and is isomorphic to a sublattice of the lattice of invariant subgroups of* $\mathbf{Q}^{\vee}$.

References

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