FREE GROUPOIDS WITH AXIOMS OF THE FORM $x^{m+1}y = xy$ and/or $xy^{n+1} = xy$

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Abstract

Main result of the paper is a canonical description of free objects in the variety $\mathcal{U}(M;N)$ of groupoids with the following axioms:

$$\{x^{m+1}\cdot y = xy|\ m\in M\}\cup\{x\cdot y^{n+1} = xy|\ n\in N\},$$

where M and N are sets of positive integers, such that $M\cup N\neq\emptyset$. Applying the obtained description, corresponding characterization of free subgroupoids of a $\mathcal{U}(M;N)$–free groupoid is given.

1 Main Results

Throughout the paper $\mathbf{F} = (F;\cdot)$ denotes the absolutely free groupoid (i.e. free groupoid in the variety of all groupoids) with a given basis B. Therefore, $\mathbf{F}$ is injective[1] and B is the set of primes[2] in F. Moreover, each subgroupoid of $\mathbf{F}$ is free and there exist subgroupoids of $\mathbf{F}$ with infinite basis (see [1],I.1)

There exist $\frac{(2k-2)!}{(k-1)!k!}$ k–th groupoid powers[3] $x\mapsto x^k$. In this paper x^k is defined by

$$x^1 = x,\quad x^{k+1} = x^kx,$$

and this is the meaning of the groupoid power in the axioms of $\mathcal{U}(M;N)$.

If $\xi,\ \eta : F\to F$ are two transformations on F, then we denote by $\mathbf{F}(\xi,\eta)$ the groupoid $(F,\bullet)$ defined by $x\bullet y = \xi(x)\eta(y)$. We say that the pair $\xi,\ \eta$ of transformations on F is *compatible* with $\mathbf{F}$ iff the following two conditions are satisfied:

1) $(\forall b\in B)\ \xi(b) = b = \eta(b)$

2) The least subset R of F with the following property:

$$B\subseteq R\ \&\ (\forall t,u\in R)\big(\xi(t) = t,\eta(u) = u\Rightarrow tu\in R\big) \tag{1}$$

is a subgroupoid of $\mathbf{F}(\xi,\eta)$.

[1] A groupoid $\mathbf{G}$ is *injective* iff $(\forall x,y,u,v\in G)(xy = uv\Rightarrow x = u\ \&\ y = v)$

[2] an element $a\in G$ is *prime* in $\mathbf{G}$ iff $a\in G\setminus GG$.

[3] see [3], III.2, Ex.2,p.125 or [8],pp.39-40

Here we introduce several notations.

The varieties $\mathcal{U}(M;\emptyset)$, $\mathcal{U}(\emptyset;N)$, $\mathcal{U}(M;N)$, where $M \neq \emptyset$ and $N \neq \emptyset$, are said to be *left, right, two–sided*, respectively. The variety $\mathcal{U}(M;\emptyset)$ will be also denoted by $\mathcal{U}_l(M)$, and $\mathcal{U}(\emptyset;N)$ by $\mathcal{U}_r(N)$. Further on, $\mathcal{U}(m_1, m_2, \cdots; n_1, n_2, \cdots)$ will be an abbreviation for $\mathcal{U}(\{m_1, m_2, \cdots\}; \{n_1, n_2, \cdots\})$

We state below the main results of the paper.

Theorem 1 *If B is a nonempty set and M, N sets of positive integers such that $M \cup N \neq \emptyset$, then there exists a pair (ξ, η) of transformations on F compatible with $\mathbf{F}$ with the following properties:*

- (i) *The restrictions of ξ and η on R are retractions of R.*
- (ii) *The corresponding groupoid $\mathbf{R}$ is a $\mathcal{U}(M;N)$–free groupoid with a unique basis B, B being the set of primes in $\mathbf{R}$.*

(We say that $\mathbf{R}$ is the $\mathcal{U}(M;N)$–*canonical groupoid* with the basis B.)

Theorem 2 *The class of free objects in a variety $\mathcal{U}(M;N)$ is hereditary iff*

$$(M \neq \emptyset,\ N = \emptyset) \text{ or } (M = \emptyset,\ 1 \in N).$$

Theorem 3 *Let $\mathbf{H}$ be a $\mathcal{U}(M;N)$–free groupoid with the basis B. If B contains at least two distinct elements or $\mathcal{U}(M;N) \notin \{\mathcal{U}_l(1), \mathcal{U}_r(1)\} \cup \{\mathcal{U}(m;1) : m \geq 1\}$, then there exists $\mathcal{U}(M;N)$–free subgroupoid of $\mathbf{H}$ with infinite basis.*

In Section 2 we state some preliminary results, and in Section i+2 we give the proof of Theorem i. Moreover, in Section 4 we describe the family of free subgroupoids of a $\mathcal{U}(M;N)$–free groupoid, in the case when the class of $\mathcal{U}(M;N)$-free groupoids is not hereditary.

2 Preliminaries

Here we state some properties of the groupoid $\mathbf{F}$ and one of the main results of [6]. Let $x \mapsto |x|$ be the homomorphism of $\mathbf{F}$ into the additive groupoid of positive integers which extends the mapping $B \to \{1\}$. In other words, we have:

$$\begin{aligned} &(\forall b \in B)\ |b| = 1, \\ &(\forall x, y \in F)\ |xy| = |x| + |y|. \end{aligned} \tag{2}$$

(We say that $|t|$ is the *length* of t in $\mathbf{F}$.)

Below we assume that m is a given positive integer, $p,\ q$ arbitrary nonnegative integers, and $i, j, k, \cdots$ arbitrary positive integers. We define two kinds of groupoid powers $x \mapsto x^{(p)},\ x \mapsto x^{\langle p\rangle}$ as follows:

$$x^{(0)} = x^{\langle 0\rangle} = x,\ x^{(p+1)} = (x^{(p)})^{m+1};\ x^{\langle p+1\rangle} = x\ \underline{x^{\langle p\rangle} m}, \tag{3}$$

where the right-hand side of the last equation has the following meaning:

$$x\ \underline{y0} = x,\ x\ \underline{yp+1} = (x\ \underline{yp})y. \tag{4}$$

By induction on the length of elements of F we obtain that, for any $t, u \in F,\ p, q \geq 0,\ i, j \geq 1$, the following relations hold:

$$|t^i| = i|t|;\ |t^{(p)}| = (m+1)^p|t|;\ |t^{\langle p\rangle}| = |t|\sum_{q=0}^{p} m^q; \tag{5}$$

$$t^{i+1} = u^{j+1} \Rightarrow t = u,\ i = j; \tag{6}$$

$$t^{(p)} = u^{(p+q)} \iff t = u^{(q)}; \tag{7}$$

$$(t^{(p)})^{(q)} = t^{(p+q)};$$

$$1 \leq i < m \Rightarrow \left(t^{i+1} \neq u^{(p+1)}\ \&\ t\ \underline{t^{\langle p\rangle} i} \neq u^{\langle q+1\rangle}\right); \tag{8}$$

$$t^{\langle p+1\rangle} = u^{\langle q+1\rangle} \iff t = u,\ p = q. \tag{9}$$

One of the main results in [6] is the following:

Theorem 2.1 *If M and N are nonempty sets of positive integers, then:*

(*i*) $\mathcal{U}(M;\emptyset) = \mathcal{U}(\mathrm{gcd}(M);\emptyset)$;[4]

(*ii*) $\mathcal{U}(\emptyset;N) = \mathcal{U}(\emptyset;\langle N\rangle)$; [5]

(*iii*) $\mathcal{U}(M;N) = \mathcal{U}(\mathrm{gcd}(M);\mathrm{gcd}(M \cup N))$.□

Considering Theorem 2.1 we shall examine three types of $\mathcal{U}(M;N)$ varieties with corresponding canonical sets of axioms, i.e. $\mathcal{U}(\emptyset;S)$,$\mathcal{U}(m;\emptyset)$ and $\mathcal{U}(m;n)$ which will be denoted as $\mathcal{U}_r(S)$,$\mathcal{U}_l(m)$ and $\mathcal{U}(m;n)$ - respectively. Here S is the additive groupoid of positive integers generated by N, $m = \mathrm{gcd}(M)$ and $n = \mathrm{gcd}(M \cup N)$in the case when both $M \neq \emptyset$ and $N \neq \emptyset$.

[4] $\mathrm{gcd}(M)$ denotes the greatest commondivisor of M.

[5] $\langle N\rangle$ is the subgroupoid of the additive groupoid of positive integers generated by N.

We shall also use the following relations:[6]

$$\mathcal{U}_l(m) \models x^{(p)}y = xy; \tag{10}$$

$$\mathcal{U}_l(m) \models x^{(p)} = x^{\langle p\rangle}; \tag{11}$$

$$\mathcal{U}_l(m) \models x^{pm+i+1} = x^{i+1}; \tag{12}$$

$$\mathcal{U}_r(i) \models (x^{i+1})^{j+1} = x^{i+j+1}; \tag{13}$$

$$\mathcal{U}(kn;n) \models (x^{in+1})^{kn+1} = x^{in+1}. \tag{14}$$

3 $\mathcal{U}(M;N)$–Canonical Groupoids

We assume below that m is a positive integer, and S is an additive groupoid of positive integers.

Define two transformations $\alpha, \beta : F \to F$, as follows:

$$\alpha(u) = \begin{cases} t, & \text{if } \ u = t^{\langle p+1\rangle}, p \geq 0 \\ u, & \text{otherwise} \end{cases} \tag{15}$$

$$\beta(u) = \begin{cases} t, & \text{if } \ u = t^{i+1}, i \in S \\ u, & \text{otherwise} \end{cases} \tag{16}$$

By (2.8) and (2.5), α and β are well defined.

Assume now that M and N are sets of positive integers such that $M \cup N \neq \emptyset$. Using α and β, we define two transformations$\xi,\ \eta : F \to F$ for each of the following cases $\mathcal{U}_l$,$\mathcal{U}_r$,$\mathcal{U}$:

$\mathcal{U}_l$: If $M \neq \emptyset,\ N = \emptyset,\ m = \gcd(M)$, then$\xi = \alpha$ and $\eta = 1_F$;

$\mathcal{U}_r$: If $M = \emptyset,\ N \neq \emptyset,\ S = \langle N\rangle$, then $\xi = 1_F$ and $\eta = \beta$;

$\mathcal{U}$: If $M \neq \emptyset,\ N \neq \emptyset,\ m = \gcd(M),\ n = \gcd(M \cup N),\ S = \{in : i \geq 1\}$, then $\xi = \alpha$ and $\eta = \beta$.

Clearly, in each of the cases: $\mathcal{U}_l$, $\mathcal{U}_r$, $\mathcal{U}$ the condition 1) of Section 1 (for the pair (ξ, η) to be compatible with $\mathbf{F}$) is satisfied. Moreover, according to the condition 2), the corresponding subset R of F is defined as follows:

$$B \subseteq R \text{ and}$$

$$\begin{array}{rc} \mathcal{U}_l : & (\forall v, w \in F)\big(vw \in R \iff v, w \in R \ \&\ \alpha(v) = v\big) \\ \mathcal{U}_r : & (\forall v, w \in F)\big(vw \in R \iff v, w \in R \ \&\ \beta(w) = w\big) \\ \mathcal{U} : & (\forall v, w \in F)\big(vw \in R \iff v, w \in R \ \&\ \alpha(v) = v \ \&\ \beta(w) = w\big). \end{array} \tag{17}$$

[6] $\mathcal{V} \models \tau_1 = \tau_2$ means: the equation $\tau_1 = \tau_2$ is true in the variety $\mathcal{V}$.

From (3.1), (3.2) and (3.3), we obtain the following relations:

$$
\begin{array}{ll}
\mathcal{U}_l: & v = u^{\langle p+1\rangle} \Rightarrow \big(v \in R \iff u \in R \ \&\ \alpha(u) = u\big); \\
\mathcal{U}_r: & v = u^{i+1} \Rightarrow \big(v \in R \iff u \in R \ \&\ \beta(u) = u\big); \\
\mathcal{U}: & v = u^{\langle p+1\rangle} \Rightarrow \big(v \in R \iff u \in R, p = 0 \ \&\alpha(u) = u\big); \\
& i \geq 1, v = u^{in+1} \Rightarrow \big(v \in R \iff u \in R, i \leq k \ \&\beta(u) = u\big), \text{ where } kn = m.
\end{array} \tag{18}
$$

From (3.4), we obtain:

Proposition 3.1 *The restrictions of ξ and η on R are retractions of R.* $\square$

From the definition of the groupoid $\mathbf{F}(\xi, \eta)$ and Proposition 3.1 it follows:

Proposition 3.2 $\mathbf{R} = (R, \bullet)$ *is a subgroupoid of $\mathbf{F}(\xi, \eta)$, and B is the least generating subset of R.* $\square$

From (3.4), the definitions of the pair (ξ, η) and Proposition 3.1 it follows that for each $u \in R$, there exists a unique $t \in R$ and a unique:
$p \geq 0$, in the case $\mathcal{U}_l$; $i \in S \cup \{0\}$, in the case $\mathcal{U}_r$; $s: \ 0 \leq s \leq k$, in the case $\mathcal{U}$, such that:

$$
\mathcal{U}_l : u = t^{\langle p\rangle}, \ \alpha(t) = t; \ \mathcal{U}_r : u = t^{i+1}, \ \beta(t) = t; \ \mathcal{U} : u = t^{sn+1}, \ \beta(t) = t. \tag{19}
$$

If $v, w \in R$, then $v \bullet w$ can be expressed more explicitely as follows:

$$
\begin{array}{ll}
\mathcal{U}_l: & v \bullet w = tw, \text{ where } v = t^{\langle p\rangle}, p \geq 0, \alpha(t) = t; \\
\mathcal{U}_r: & v \bullet w = vu, \text{ where } w = u^{i+1}, i \in S \cup \{0\}, \beta(u) = u; \\
\mathcal{U}: & v \bullet w = \begin{cases} vu, & \text{if } \alpha(v) = v, \ w = u^{in+1}, 1 \leq i \leq k \\ tw, & \text{if } v = t^{m+1}, \beta(w) = w \\ tu, & \text{if } v = t^{m+1}, w = u^{in+1}, 1 \leq i \leq k \\ vw, & \text{if } \alpha(v) = v, \beta(w) = w. \end{cases}
\end{array} \tag{20}
$$

Now we shall show the following:

Proposition 3.3 $\mathbf{R} \in \mathcal{U}(M; N)$.

Proof. If $u \in R, j \geq 1$, then we denote by $u_\bullet^j$ the j–th power of u in $\mathbf{R}$, i.e.

$$
u_\bullet^1 = u, \quad u_\bullet^{j+1} = (u_\bullet^j) \bullet u. \tag{21}
$$

(Note that, if $u \in R$, then $u_\bullet^j \in R$, but it can happen that $u^j \in F \setminus R$.)
Assuming (3.5), by (3.6) we obtain the equalities (3.8) in the corresponding cases $\mathcal{U}_l$, $\mathcal{U}_r$ and $\mathcal{U}$.

$$
\begin{aligned}
\mathcal{U}_l:&\qquad u_\bullet^j = \begin{cases} u, & \text{if } j=1 \\ t\ \underline{uj-1}, & \text{if } 2 \le j \le m \\ t\ \underline{ur-1}, & \text{if } j = qm+r, 2 \le r \le m+1 \end{cases} \\
\mathcal{U}_r:&\qquad u_\bullet^j = t^{i+j},\ \text{if } j \ge 1; \\
\mathcal{U}:&\quad u_\bullet^j = \begin{cases} u, & \text{if } j=1 \\ t^{sn+j}, & \text{if } s \ge 1, 1 \le j \le (k-s)n+1. \\ t^{j-(k-s)n}, & \text{if } s \ge 1, (k-s)n+2 \le j \le kn+1 \end{cases}
\end{aligned}
\tag{22}
$$

Therefore:

$$
\begin{aligned}
\mathcal{U}_l:&\qquad u_\bullet^{m+1} = t^{\langle p+1\rangle}; \\
\mathcal{U}_r:&\qquad u_\bullet^{j+1} = u^{i+j+1},\quad \text{for each } j \in S; \\
\mathcal{U}:&\quad u_\bullet^{in+1} = \begin{cases} t^{(i+s)n+1}, & \text{if } i+s \le k \\ t^{(i+s-k)n+1}, & \text{if } i+s > k, 1 \le i \le k. \end{cases}
\end{aligned}
\tag{23}
$$

If $\eta(u) = t \neq u$ (i.e. $u = t^{sn+1},\ s \ge 1$) and $i = k$, then in the last case we obtain

$$u_\bullet^{m+1} = u.$$

From (3.6) and (3.9) we obtain that, for any $v, w \in R$, the following equations hold:

$\mathcal{U}_l\ :\ (v_\bullet^{m+1}) \bullet w = v \bullet w;$

$\mathcal{U}_r\ :\ v \bullet (w_\bullet^{i+1}) = v \bullet w$, for each $i \in S$;

$\mathcal{U}\ :\ (v_\bullet^{m+1}) \bullet w = v \bullet w = v \bullet (w_\bullet^{n+1}).$

Therefore, we have $\mathbf{R} \in \mathcal{U}_l(m)$, $\mathbf{R} \in \mathcal{U}_r(S)$, $\mathbf{R} \in \mathcal{U}(kn;n)$ in the cases: $\mathcal{U}(M;N) = \mathcal{U}_l(m)$, $\mathcal{U}(M;N) = \mathcal{U}_r(S)$, $\mathcal{U}(M;N) = \mathcal{U}(kn;n)$–respectively. □

The following statement will complete the proof of Theorem 1.

Proposition 3.4 *Let $\mathbf{G} = (G;\cdot) \in \mathcal{U}(M;N)$. If $\lambda : B \to G$ is a mapping, and $\varphi : \mathbf{F} \to \mathbf{G}$ the homomorphism which extends λ, then the restriction of φ on R is a homomorphism from $\mathbf{R}$ into $\mathbf{G}$.*

Proof. It suffices to show the equality $\varphi(v \bullet w) = \varphi(v)\varphi(w)$, for each $v, w \in R$ such that $v \bullet w \neq vw$.

Then, in the case $\mathcal{U}(M;N) = \mathcal{U}_l(m)$, we have $v = t^{\langle p+1\rangle}$ for a unique pair (t,p), where $t \in R, p \ge 0, \alpha(t) = t$, and $v \bullet w = tw$. Therefore, we have:

$$\varphi(v \bullet w) = \varphi(tw) = \varphi(t)\varphi(w) = \varphi(t)^{(p+1)}\varphi(w).$$

Then, by (2.10) we have:

$$\varphi(t)^{(p+1)}\varphi(w) = \varphi(t)^{\langle p+1\rangle}\varphi(w) = \varphi(t^{\langle p+1\rangle})\varphi(w) = \varphi(v)\varphi(w).$$

In the case $\mathcal{U}(M;N) = \mathcal{U}_r(S)$ we have: $w = t^{i+1}$, for a unique pair (t,i), where $t \in R, \beta(t) = t, i \in S$. Therefore,

$$\varphi(v \bullet w) = \varphi(vt) = \varphi(v)\varphi(t) = \varphi(v)\varphi(t)^{i+1} = \varphi(v)\varphi(t^{i+1}) = \varphi(v)\varphi(w).$$

In a similar way, we obtain that $\varphi(v \bullet w) = \varphi(v)\varphi(w)$, in the case $\mathcal{U}$. □

By Proposition 3.2 – Propoosition 3.4, **R** is a $\mathcal{U}(M;N)$–free groupoid with the unique basis B, i.e. we have completed the proof of Theorem 1.

We say that a formula: $x^{m+1} \cdot y = xy$ ($x \cdot y^{n+1} = xy$) is a *left* (a *right*) equation; a left or a right equation is called an *equation*. It is well known that an equation holds in a variety $\mathcal{U}(M;N)$ iff it is satisfied in each $\mathcal{U}(M;N)$–free groupoid. Therefore, the following statement describes the set of equations in a variety $\mathcal{U}(M;N)$.

Proposition 3.5 *Let* **H** *be a free groupoid in the variety* $\mathcal{U}(M;N)$. *Then the following statements hold.*

(*i*) *If* $M \neq \emptyset,\ N = \emptyset,\ \gcd(M) = m$, *then a left equation* $x^{m+1}y = xy$ *holds in* **H** *iff* $m|n$*; no right equation holds in* **H**.

(*ii*) *If* $M = \emptyset,\ N \neq \emptyset$, *then the right equation* $xy^{j+1} = xy$ *holds in* **H** *iff* $j \in \langle N\rangle$*; no left equation holds in* **H**.

(*iii*) *If* $M \neq \emptyset,\ N \neq \emptyset,\ \gcd(M) = m,\ n = \gcd(M \cup N)$, *then the equations* $x^{i+1}y = xy$ *and* $xy^{j+1} = xy$ *hold in* **H** *iff* $m|i$ *and* $n|j$.

Proof. Let **R** be a $\mathcal{U}(M;N)$–canonical groupoid with the basis B, and $a, b \in B$. Then:

(*i*) If $M \neq \emptyset,\ N = \emptyset,\ \gcd(M) = m$, then
$(a_\bullet^{i+1}) \bullet b = ab = a \bullet b$ iff $m|i$;
$a \bullet b^{j+1} \neq ab = a \bullet b$ for each $j \geq 1$.

(*ii*) If $M = \emptyset,\ N \neq \emptyset$ and $S = \langle N\rangle$, then:
$a^{i+1} \bullet b = a^{i+1}b \neq ab = a \bullet b$;
and, if $j \geq 1$, then $a \bullet (a_\bullet^{j+1}) = ab = a \bullet b$ iff $j \in S$.

(*iii*) If $M \neq \emptyset,\ N \neq \emptyset,\ \gcd(M) = m,\ n = \gcd(M \cup N),\ i, j \geq 1$, then:
$(a_\bullet^{i+1}) \bullet b = ab = a \bullet b$ iff $m|i$,
$a \bullet (b_\bullet^{j+1}) = ab = a \bullet b$ iff $n|j$. □

Having in mind the definitions of the transformations ξ, η in each of the cases $\mathcal{U}_l(m), \mathcal{U}_r(S)$ and $\mathcal{U}(kn; n)$, as a corollary of Theorem 1 the following statement can be also obtained.

Proposition 3.6 *If* $\mathbf{H}$ *is a* $\mathcal{U}(M; N)$*–free groupoid with the basis* B*, then there exist retractions* γ *and* δ *of* H *with the following properties:*

- (i) *B is the set of primes in* $\mathbf{H}$*, and* $B \subseteq \mathrm{im}\gamma \cap \mathrm{im}\delta$*;*

  (If $x \in \mathrm{im}\gamma \cap \mathrm{im}\delta$, then we say that x is a *base* in $\mathbf{H}$)

- (ii) $(\forall x, y \in H) xy = \gamma(x)\delta(y)$*;*

  $((\gamma(x), \delta(y))$ is the *pair of divisors* of xy in $\mathbf{H}$; i.e. $\gamma(x)$ is the *left* and $\delta(y)$ the *right divisor* of xy.)

- (iii) *There exists a mapping* $x \mapsto |x|$ *from* H *into the set of positive integers with the following properties:*

$$|xy| = |\gamma(x)| + |\delta(y)|,$$

$$\gamma(x) \neq x \iff |\gamma(x)| < |x|; \ \delta(x) \neq x \iff |\delta(x)| < |x|,$$

  for any $x, y \in H$.

Proof. If $\mathbf{R}$ is the $\mathcal{U}(M; N)$–canonical groupoid with the basis B, then there exists a unique isomorphism $\varphi : \mathbf{R} \to \mathbf{H}$ such that $\varphi(b) = b$, for each $b \in B$. Defining $\gamma, \delta : H \to H$ by: $\gamma(x) = \xi(\varphi^{-1}(x)),\ \delta(x) = \eta(\varphi^{-1}(x))$, we obtain two retractions γ, δ of H such that (i)– (iii) hold, where the length of $x \in H$ is defined by $|x| = |\varphi^{-1}(x)|$. □

In each of the cases $\mathcal{U}_l(m), \mathcal{U}_r(S), \mathcal{U}(kn; n)$, the results of Proposition 3.6 can be stated more explicitely as follows.

3.6. $\mathcal{U}_l(m)$.

- (i) $\gamma = \alpha, \delta = 1_H$;

- (ii) $y \in H$ is a *base in* $\mathbf{H}$ iff $y \in \mathrm{im}\gamma$; for each $x \in H$ there exists a unique $y = \mathrm{bs}(x)$ (the *base of* x) and unique $p = \exp(x) \geq 0$ (the *exponent of* x) suchthat $x = y^{(p)}$.

- (iii) $\mathrm{bs}(x)$ is the left (and y the right) divisor of xy.

- (iv) If b is a base in $\mathbf{H}$, and $1 \leq i < m,\ p \geq 0$,then $c = b\ \underline{b^{(p)}i}$ is also a base in H; $b\ \underline{b(p)i-1}$ is the left and $b^{(p)}$ the right divisor of c; in the same case $b\ \underline{b^{(p)}m-1}$ is the left and $b^{(p)}$ the right divisor of $b^{(p+1)}$.

- (v) If $x \in H, 1 \leq i \leq j \leq m+1$, then $x^i = x^j \Rightarrow i = j$.

3.6. $\mathcal{U}_r(N)$.

(i) $\gamma = 1_H, \delta = \beta$;

(ii) y is a base in **H** iff $y \in \mathrm{im}\delta$; for each $x \in H$ there exists a unique base y, and a unique $q \in \{0\} \cup \langle N \rangle$, such that $x = y^{q+1}$.

(iii) The left divisor of xy is x and its right divisor is $\mathrm{bs}(y)$. Thus,

$$xy = uv \iff x = u,\ \mathrm{bs}(y) = \mathrm{bs}(v).$$

3.6. $\mathcal{U}(kn; n)$.

(i) x is a base in **H** iff $x^{m+1} \neq x$; for each $x \in H$ there exists a unique y (the *base of* x) and a unique $i \in \{0, 1, \cdots, k\}$ (the *exponent of* x) such that $x = y^{in+1}$; x is a *left base* in **H** iff $i \neq k$, where i is the exponent of x.

(ii) For any $x \in H$, $\delta(x)$ is the base of x, and

$$\gamma(x) = \begin{cases} x, & \text{if } x \text{ is a left base} \\ y, & \text{if } y \text{ is the base of } x, \text{ and } x = y^{m+1} \end{cases}$$

$$\delta(x) = \begin{cases} y, & \text{if } x = y^{in+1},\ 0 \leq i \leq k, \\ x, & \text{otherwise.} \end{cases}$$

(iii) $\gamma(x)$ is the left and $\delta(y)$ the right divisor of xy.

4 Free Subgroupoids of $\mathcal{U}(M; N)$–Free Groupoids

We shall describe the set of pairs (M, N) of sets of positive integers such that the class of free objects in the variety $\mathcal{U}(M; N)$ is hereditary, i.e. we shall prove Theorem 2.

Proposition 4.1 *For any $m \geq 1$, theclass of free objects un the variety $\mathcal{U}_l(m)$ is hereditary.*

Proof. Let **Q** be a subgroupoid of a $\mathcal{U}_l(m)$–free groupoid **H**. We have to show that the set P of prime elements of Q is nonempty, and that **Q** is $\mathcal{U}_l(m)$–free with the basis P. The proof will be given in several steps, where induction on $|x|$, for $x \in Q$, will be used.

1) If $a \in Q \setminus P$ and c is the right divisor of a in **H**, then $c \in Q$.

2) Let $a = b^{(p)} \in Q$, where b is the base of a in **H**. If q is the least non negative integer such that $b' = b^{(q)} \in Q$, then we say that b' is the base of a in **Q**. Then, if $q \geq 1, b' \in P$.

By 1) and 2) we obtain:

3) $P \neq \emptyset$, and P is the least generating subset of $\mathbf{Q}$.

4) If $c, d \in Q$, and b' is the base of c in $\mathbf{Q}$, then we say that (b', d) is *the pair of divisors* of cd in $\mathbf{Q}$. Then: $|d| < |cd|$, and: b' is prime or $|cd| = |b'| + |d|$.

5) Assume that $\mathbf{G} \in \mathcal{U}_l(m)$ and $\lambda : P \to G$ is a given mapping. There is a (unique) homomorphism $\varphi : Q \to G$ such that $\lambda = \varphi_{|P}$ is the restriction of φ on P. Namely, if $x \in Q$ is such that $|x| = \min\{|y| : y \in Q\}$, then $y \in P$, and thus $\varphi(x) = \lambda(x)$ is well defined. Assume that for each $x \in Q$, such that $|x| \leq i$, $\varphi(x) \in G$ is well defined and, moreover, if (y, z) is the pair of divisors of x in $\mathbf{Q}$, then $\varphi(y)$, $\varphi(z)$ are well defined, and $\varphi(x) = \varphi(y)\varphi(z)$.

Let $v \in Q \setminus P$ be such that $|v| = i + 1$, and (t, u) be the pair of divisors of v in $\mathbf{Q}$. Then $\varphi(t)$ and $\varphi(u)$ are well defined, and thus we can define $\varphi(v)$ by $\varphi(v) = \varphi(t)\varphi(u)$. Then $\varphi : \mathbf{Q} \to \mathbf{G}$ is a homomorphism which extends λ. □

Proposition 4.2 *The class of free objects in the variety $\mathcal{U}_r(1)$ is hereditary.*

Proof. This statement is one of the main results of [4], and it is also a corollary of Proposition 4.1. Namely, let $\mathbf{G} = (G, \cdot)$ be a given groupoid, and the groupoid $\mathbf{G^{op}} = (G, \circ)$ be defined by $x \circ y = yx$. Then, $\mathbf{G} \in \mathcal{U}_r(1) \iff \mathbf{G^{op}} \in \mathcal{U}_l(1)$, and $\mathbf{H}$ is $\mathcal{U}_r(1)$–free iff $\mathbf{H^{op}}$ is $\mathcal{U}_l(1)$–free. □

Proposition 4.3 *If N is a nonempty set of positive integers and $1 \notin N$, then the class of free objects in the variety $\mathcal{U}_r(N)$ is not hereditary.*

Proof. Let $n = \min(N)$, and let $\mathbf{H}$ be a $\mathcal{U}_r(N)$–free groupoid with the basis B. Consider the subgroupoid $\mathbf{Q}$ generated by $\{b^n, b^{n+1}\}$, where $b \in B$. Then b^n is the unique prime in $\mathbf{Q}$, and $\{b^n\}$ does not generate $\mathbf{Q}$, which implies that $\mathbf{Q}$ is not free [7]. □

Proposition 4.4 *If $M \neq \emptyset, N \neq \emptyset$, then the class of free objects in the variety $\mathcal{U}(M; N)$ is not hereditary.*

Proof. Let $m = \gcd(M), n = \gcd(M \cup N)$ and let $\mathbf{H}$ be a $\mathcal{U}(M; N)$–free groupoid with the basis B. If $b \in B$, and $\mathbf{Q}$ is the subgroupoid generated by $\{b^{n+1}\}$, then the set of primes in $\mathbf{Q}$ is empty. (Namely, $(b^{n+1})^{m+1} = b^{n+1}$, which implies that b^{n+1} is not a prime in $\mathbf{Q}$.) □

Theorem 2 is a corollary of Proposition 4.1–Proposition 4.4.

Proposition 4.5 *Let $\mathbf{H}$ be a $\mathcal{U}(M; N)$–free groupoid and $\mathbf{Q}$ a subgroupoid of $\mathbf{H}$, such that:*

$$(\forall x \in H)\big(x \in Q \Rightarrow \mathrm{bs}(x) \in Q\big). \tag{24}$$

Then $\mathbf{Q}$ is free.

[7] Here, and further on in Section 4, if $\mathbf{H}$ is $\mathcal{U}(M; N)$–free groupoid, and $\mathbf{Q}$ is a subgroupoid of $\mathbf{H}$, we will write "$\mathbf{Q}$ is free" instead of "$\mathbf{Q}$ is $\mathcal{U}(M; N)$–free".

Proof. From (4.1) it follows that if $a \in QQ$ and (c,d) is the pair of divisors of a in $\mathbf{H}$, then $c, d \in Q$, and moreover the following equation holds:

$$|a| = |c| + |d|.$$

This implies that the set P of primes in $\mathbf{Q}$ is nonempty and generates $\mathbf{Q}$. In the same way as 5) in the proof of Proposition 4.1, one can show that $\mathbf{Q}$ is free with the basis P. $\Box$

In the next three statements we describe free subgroupoids of $\mathcal{U}(M;N)$– free groupoids when the class of $\mathcal{U}(M;N)$–free groupoids is not hereditary.

Proposition 4.6 *Let* $\mathbf{H}$ *be a* $\mathcal{U}(M;N)$*–free groupoid, where* $M \neq \emptyset, N \neq \emptyset$, *and* $\mathbf{Q}$ *be a subgroupoid of* $\mathbf{H}$. *If* $\mathbf{Q}$ *does not satisfy* (4.1)*, then* $\mathbf{Q}$ *is not free.*

Proof. Let $m = \gcd(M), n = \gcd(M \cup N), m = kn$, and let $a \in Q$ be such that $b \notin Q$, where b is the base of a in $\mathbf{H}$. Then, there exists an $i \in \{1, 2, \cdots, k\}$, such that $a = b^{in+1}$. Then:

$$a^2 = b^{in+2},\ a^3 = b^{in+3}, \cdots,\ a^{(k-i)n+1} = b^{kn+1} = b^{m+1},$$

$$a^{(k-i)+2} = b^2,\ a^{(k-i)n+3} = b^3, \cdots, a^{(k-i)n+n} = b^n,$$

$$a^{(k-i+1)n+1} = b^{n+1}, \cdots, a^{(k-i+2)n+1} = b^{2n+1}, \cdots, a^{kn} = b^{in},\ a^{kn+1} = a$$

are elements of Q. Thus, $b^2, b^3, \cdots, b^{m+1} \in Q$, but $b \notin Q$. From the equality $(b^{m+1})^{m+1} = b^{m+1}$ it follows that b^{m+1} is not a base, and if $\mathbf{Q}$ were free, a base c in $\mathbf{Q}$ and $j \in \{1, 2, \cdots, k\}$ would exist, such that $b^{m+1} = c^{in+1}$, which would imply $i = k, c = b \in Q$, i.e. we would obtain a contradiction. $\Box$

Proposition 4.7 *Let* $\mathbf{Q}$ *be a subgroupoid of a* $\mathcal{U}_r(N)$*–free groupoid, where* $n = \min(N) \in N$, *and let, for* $x \in Q$, $j(x)$ *be defined as follows:*

$$j(x) = \min\{s : (\mathrm{bs}(x))^{s+1} \in Q, s \geq 0\}. \tag{25}$$

Then, $\mathbf{Q}$ *is free iff*

$$(\forall x \in Q)\ n | j(x). \tag{26}$$

Proof. Assuming the condition (4.3), in the same way as in the proof of Proposition 4.1, one can show that $\mathbf{Q}$ is free.

Thus, we can assume that there exists an $a \in Q$, such that n is not a divisor of $j = j(a)$. Then, b^{j+1} is a prime in $\mathbf{Q}$, where $b = \mathrm{bs}(a)$ is the base of a; moreover, then $b^{j+\nu} \in Q$, for any $\nu \geq 1$. Denote by $\mathbf{T}$ the subgroupoid of $\mathbf{Q}$ generated by the set P of primes in $\mathbf{Q}$. Thus: $T = \bigcup\{P_\nu : \nu \geq 1\}$, where $P_1 = P,\ P_{\nu+1} = P_\nu \cup \{xy : x, y \in P_\nu\}$. Then, if $i \geq 1$ is such that $in > j$, $b^{in+1} \in Q \setminus T$, and therefore the set P of primes in $\mathbf{Q}$ does not generate $\mathbf{Q}$. $\Box$

Proposition 4.8 *Let* $\mathbf{H}$ *be a* $\mathcal{U}_r(N)$*–free groupoid, where* $\gcd(N) \notin N$. *A subgroupoid* $\mathbf{Q}$ *of* $\mathbf{H}$ *is free iff* $\mathbf{Q}$ *satisfies the condition* (4.1).

Proof. Denote by S the additive groupoid of positive integers generated by N. Then $n = \gcd(S) = \gcd(N) \notin S$. If $\mathbf{Q}$ satisfies the condition (4.1), then, by Proposition 4.5, $\mathbf{Q}$ is $\mathcal{U}_r(N)$–free.

Assume that there exists an $a \in Q$, such that $b \notin Q$, where b is the base of a in $\mathbf{H}$. Therefore, there exists an $s \in S$, such that $a = b^{s+1}$. Let

$$i = \min\{\nu : b^{\nu+1} \in Q, \nu \geq 1\}. \tag{27}$$

Then $i \geq 1$, and $b^{i+1} \in Q$; moreover b^{i+1} is prime in $\mathbf{Q}$. If $i \notin S$, then one can show that $\mathbf{Q}$ is not $\mathcal{U}_r(N)$–free in the same way as in the Proposition 4.7, in the case "n is not a divisor of j".

Thus we can assume that $i \in S$. Then $b^{i+\nu} \in Q$, for any $\nu \geq 1$.

Let s be the least element of S such that $s + \nu n \in S$, for any $\nu \geq 1$. (See [6, Lemma 1.6.iii] or [7].) Then $j = s + (i - n) \in S$, but $j - i = s - n \notin S$. Let k be the least element of S such that $i < k \leq j$, and $j - k \in S \cup \{0\}$. Then: $i + j = k + i + \alpha$, where $\alpha \in S \cup \{0\}$.

Assume that $\mathbf{Q}$ is free. Then b^{i+1} and b^{k+1} are different bases in $\mathbf{Q}$, but

$$(b^{i+1})^{j+1} = b^{i+j+1} = b^{i+k+\alpha+1} = (b^{k+1})^{i+\alpha+1},$$

which is impossible. □

5 Ranks of Free Subgroupoids of $\mathcal{U}(M;N)$–free Groupoids

We shall first consider $\mathcal{U}(M;N)$-free groupoids with one element basis $B = \{b\}$ in the cases $\mathcal{U}_l(1)$, $\mathcal{U}_r(1)$ and $\mathcal{U}(1;1)$ and then prove Theorem 3.

Proposition 5.1 *If Z^+ is the set of positive integers, then the groupoid $(Z^+, \bullet)$ defined by $i \bullet j = i + 1$ is $\mathcal{U}_l(1)$–free groupoid with the basis $\{1\}$. If $\mathbf{Q}$ is a subgroupoid of $(Z^+, \bullet)$ and m is the least element of Q, then $\mathbf{Q}$ is a $\mathcal{U}_l(1)$–free groupoid with the basis $\{m\}$. The groupoid $(Z^+, \bullet)^{op}$ is $\mathcal{U}_r(1)$–free with the basis $\{1\}$.* □

Proposition 5.2 *The groupoid $\mathbf{H} = (\{1, 2, \cdots, m, m+1\}, \bullet)$, defined by*

$$i \bullet j = \begin{cases} i + 1, & \text{for } i \leq m \\ 2, & \text{for } i = m + 1 \end{cases}$$

is $\mathcal{U}(1;1)$–free groupoid with the basis $\{1\}$. If Q is a proper subgroupoid of $\mathbf{H}$, then $\mathbf{Q}$ is not $\mathcal{U}(1;1)$–free.

Proof. $Q = \{2, 3, \cdots, m, m+1\}$ is the unique proper subgroupoid in $\mathbf{H}$. The set of primes in $\mathbf{Q}$ is empty, and, thus, $\mathbf{Q}$ is not $\mathcal{U}(1;1)$–free. □

Proposition 5.3 *Let* $\mathbf{H}$ *be a* $\mathcal{U}(M;N)$*–free groupoid with the basis* $\{b\}$*, where*

$$\mathcal{U}(M;N) \notin \{\mathcal{U}_l(1), \mathcal{U}_r(1)\} \cup \{\mathcal{U}(m;1) : m \geq 1\}, \tag{28}$$

and let $A = \{a_i : i \geq 1\} \subseteq H$ *be defined as follows:*

$$a_1 = b^2, \quad a_{i+1} = ba_i. \tag{29}$$

If $\mathbf{Q}$ *is a subgroupoid of* $\mathbf{H}$ *generated by* A*, then* $\mathbf{Q}$ *is* $\mathcal{U}(M;N)$*–free with the basis* A*, and* $a_i = a_j \Rightarrow i = j$*.*

Proof. Assuming that $\mathbf{H} = \mathbf{R}$ is the $\mathcal{U}(M;N)$–canonical groupoid with the basis $\{b\}$, we obtain that $a_i = a_j \Rightarrow i = j$, i.e. A is an infinite subset of H. Moreover, for each $i \geq 1$, b is the left divisor for a_i, and $b \notin Q$. This implies that a_i is prime in $\mathbf{Q}$.

It remains to show that $\mathbf{Q}$ is $\mathcal{U}(M;N)$–free with the basis A.

If $\mathcal{U}(M;N) = \mathcal{U}_l(m)$, by Proposition 4.1 we obtain that $\mathbf{Q}$ is $\mathcal{U}(M;N)$–free with the basis A.

Assume now that $\mathcal{U}(M;N) = \mathcal{U}_r(S), 1 \notin S = \langle N \rangle$, and that $d = c^{i+1} \in Q$, where $i \in S$, and c is a base in $\mathbf{H}$. Then $d \notin A$, and therefore $c^i \in Q$. Continuing in such a way, we would obtain $c \in Q$. Thus by Proposition 4.5, (4.1) is satisfied, and thus $\mathbf{Q}$ is $\mathcal{U}_r(S)$–free with the basis A.

Finally, let $\mathcal{U}(M;N) = \mathcal{U}(kn;n), n \geq 2$. The fact that A generates $\mathbf{Q}$ implies:

$$Q = \bigcup \{A_i : i \geq 1\}, \text{ where } A_1 = A, \; A_{i+1} = A_i \cup \{xy : x, y \in A_i\}.$$

Assume that $d = c^{in+1} \in Q$, where $1 \leq i \leq k$, and c is a base in $\mathbf{H}$. Let s be the least positive integer, such that $d \in A_{s+1} \setminus A_s$. Such an s exists as $d \notin A$. Thus, there exist $d', d'' \in A_s$, such that $c^{in+1} = c^{in}c = d = d'd''$, and therefore $c^{in} = d'$, $d'' = c^{jn+1}$ for some $0 \leq j \leq k$. So, $c^{in} \in Q \setminus A$; then, by the same argument, $c^{in-1} \in Q$ e.t.c, and by an obvious induction we obtain that $c \in Q$. Therefore, $\mathbf{Q}$ is free. □

(We note that in the case $\mathcal{U}(M;N) = \mathcal{U}_l(m)$, $\mathbf{Q}$ satisfies the relation (4.1). Also:

$\mathcal{U}(M;N) = \mathcal{U}_l(1) \Rightarrow A = \{b^2\}$, and $\mathbf{Q}$ is $\mathcal{U}_l(1)$–free with the basis A;

$\mathcal{U}(M;N) = \mathcal{U}_r(1) \Rightarrow Q = A$, and $\mathbf{Q}$ is $\mathcal{U}_r(1)$–free with the basis $\{b^2\}$;

$\mathcal{U}(M;N) = \mathcal{U}(m;1) \Rightarrow A = \{b^2\}, Q = \{b^2, b^3, \cdots, b^{m+1}\}$, and $\mathbf{Q}$ is not $\mathcal{U}(m;1)$–free.)

The proof of the following statement is the same as the proof of Proposition 5.3, and moreover, the assumption (5.1) is not necessary.

Proposition 5.4 *Let* $\mathbf{H}$ *be a* $\mathcal{U}(M;N)$*–free groupoid with the basis* $\{a,b\}, a \neq b$*, and* $C = \{c_i : i \geq 1\}$ *be defined as follows:*

$$c_1 = ab, \quad c_{i+1} = ac_i. \tag{30}$$

Then $c_i = c_j \Rightarrow i = j$*, and the subgroupoid* $\mathbf{Q}$ *of* $\mathbf{H}$ *generated by* C *is* $\mathcal{U}(M;N)$*–free with the basis* C*.* □

This completes the proof of Theorem 3.

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