

# VARIETIES OF GROUPOIDS WITH AXIOMS OF THE FORM

$$x^{m+1}y = xy \text{ and/or } xy^{n+1} = xy$$

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## Abstract

The subject of this paper are varieties  $\mathcal{U}(M; N)$  of groupoids defined by the following system of identities

$$\{x^{m+1} \cdot y = xy : m \in M\} \cup \{x \cdot y^{n+1} = xy : n \in N\},$$

where  $M, N$  are sets of positive integers. The equation  $\mathcal{U}(M; N) = \mathcal{U}(M'; N')$  for any given pair  $(M, N)$  is solved, and, among all solutions, one called canonical, is singled out. Applying a result of Evans ([6]) it is shown for finite  $M$  and  $N$  that: if  $M$  and  $N$  are nonempty and  $\gcd(M) = \gcd(M \cup N)$ , or only one of  $M$  and  $N$  is nonempty, then the word problem is solvable in  $\mathcal{U}(M; N)$ .

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## 0 Introduction

A *groupoid* is an algebra  $\mathbf{G} = (G, \bullet)$  with one binary operation  $\bullet : (a, b) \mapsto ab$ . (We will often omit the operation sign.) Assuming the usual meanings of other algebraic notions, we do not define them explicitly.

By a result of P. Hall (see, for example, [3], III.2, Ex.2, p.125, or [10], pp 39-40), for any positive integer  $k$  there exist  $\frac{(2k-2)!}{k!(k-1)!}$   $k$ -th *groupoid powers*  $x \mapsto x^k$ . In this paper, we assume the groupoid power  $x^k$  defined as follows:

$$x^1 = x, \quad x^{k+1} = x^k x.$$

So  $x^3 = x^2 x = (xx)x$ .

A formula  $x^{k+1}y = xy$  ( $xy^{k+1} = xy$ ), will be called a *left (right) equation*. (Here, and further on,  $m, n, k, p, i, j, s$  are assumed to be positive integers, and  $xy^{n+1}$  stands for  $x \cdot y^{n+1}$ , and  $x^{m+1}y$  for  $x^{m+1} \cdot y$ .) The varieties

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$\mathcal{U}(M; \emptyset)$ ,  $\mathcal{U}(\emptyset; N)$ ,  $\mathcal{U}(M; N)$ , where  $M \neq \emptyset$  and  $N \neq \emptyset$ , are said to be *left*, *right*, *two-sided*, respectively. (Throughout the paper “variety” will mean “left, right, or two-sided variety”.)

Below,  $\mathcal{U}(m_1, m_2, m_3, \dots; n_1, n_2, n_3, \dots)$  will be an abbreviation for  $\mathcal{U}(\{m_1, m_2, m_3, \dots\}; \{n_1, n_2, n_3, \dots\})$ .

The paper consists of three sections. In Section 1 we show that each variety  $\mathcal{U}(M; N)$  admits a canonical axiom system. In Section 2 we solve the equation  $\mathcal{U}(M; N) = \mathcal{U}(M'; N')$ . Finally, in Section 3, we consider “incomplete  $\mathcal{U}(M; N)$ -groupoids”, and applying a result of Evans ([6]) we show that the word problem is solvable in  $\mathcal{U}(M; N)$  for finite  $M$  and  $N$  in each of the cases: (i)  $N = \emptyset$ , (ii)  $M = \emptyset$ , (iii)  $M \neq \emptyset$ ,  $N \neq \emptyset$ ,  $\gcd(M) = \gcd(M \cup N)$ .<sup>2</sup>

## 1 A Canonical Axiom System for $\mathcal{U}(M; N)$

The main result of this section is the following

**Theorem 1** *If  $M, N$  are nonempty sets of positive integers, then*

- (l)  $\mathcal{U}(M; \emptyset) = \mathcal{U}(\gcd(M); \emptyset)$ .
- (r)  $\mathcal{U}(\emptyset; N) = \mathcal{U}(\emptyset; \langle N \rangle)$ .<sup>3</sup>
- (t)  $\mathcal{U}(M; N) = \mathcal{U}(\gcd(M); \gcd(M \cup N))$ .

In order to prove this theorem we will show some lemmas, where  $m, n, k, p, i, j, s$  are assumed to be positive integers as above, and  $q$  a nonnegative integer.

**Lemma 1.1** *If  $1 \leq k \leq m$ , then*

$$\mathcal{U}(m; \emptyset) \models x^{qm+k+1} = x^{k+1}. \quad 4$$

**Proof.** Clearly,  $x^{m+2} = x^2, \dots, x^{2m+1} = x^{m+1}$  are true in  $\mathcal{U}(m; \emptyset)$ ; then the proof follows by induction on  $q$  and  $k$ .  $\square$

As a corollary, we obtain:

**Lemma 1.2** *If  $m|n$ , then  $\mathcal{U}(m; \emptyset) \subseteq \mathcal{U}(n; \emptyset)$ .<sup>5</sup>  $\square$*

**Lemma 1.3** *If  $\gcd(M) = d \notin M$ , then there exists a nonempty set  $M_1$  of positive integers such that*

$$\mathcal{U}(M; \emptyset) = \mathcal{U}(M_1; \emptyset), \quad d = \gcd(M_1), \quad \min(M_1) < \min(M). \quad 6 \quad (1)$$

<sup>2</sup> $\gcd(M)$  is the greatest common divisor of  $M$

<sup>3</sup> $\langle N \rangle$  is the additive groupoid of integers generated by  $N$ .

<sup>4</sup> $\mathcal{V} \models \tau_1 = \tau_2$  means: the equation  $\tau_1 = \tau_2$  is true in the variety  $\mathcal{V}$ .

<sup>5</sup> $m|n$  denotes that  $m$  is a divisor of  $n$ .

<sup>6</sup> $\min(M)$  denotes the least element in  $M$ .

**Proof.** Let  $p = \min(M)$ . The assumption  $d \notin M$  implies that  $d < p$  and thus there exists an  $n \in M$  such that  $p$  is not a divisor of  $n$ . Then  $n = qp + k$ ,  $d|k$ ,  $k < p$  and, if  $M_1 = (M \setminus \{n\}) \cup \{k\}$ , the relations (1.1) hold.  $\square$

As a corollary of Lemma 1.2 and Lemma 1.3 we obtain the equality (l).

The equality (r) is an obvious corollary of the following

**Lemma 1.4**  $\mathcal{U}(\emptyset; m, n) \subseteq \mathcal{U}(\emptyset; m + n)$ .

**Proof.**  $\mathcal{U}(\emptyset; m) \models (x^{m+1})^i = x^{m+i}$ , and therefore  $\mathcal{U}(\emptyset; m, n) \models (x^{m+1})^{n+1} = x^{m+n+1}$ . Thus, if  $\mathbf{G} \in \mathcal{U}(\emptyset; m, n)$ , then:

$$x^{m+n+1}y = (x^{m+1})^{n+1} = x^{m+1}y = xy, \text{ i.e. } \mathbf{G} \in \mathcal{U}(\emptyset; m + n). \square$$

It remains to prove (t).

**Lemma 1.5** If  $L = \{\gcd(m, n) : n \in N\}$ , then  $\mathcal{U}(m; N) = \mathcal{U}(m; L)$ .

**Proof.** By a similar argument as in Lemma 1.2,  $\mathcal{U}(m; L) \subseteq \mathcal{U}(m; N)$ . If  $n \in N$  and  $d = \gcd(m, n)$ , then there exist  $i, j$  such that  $im + d = jn$ . By Lemma 1.1,  $\mathcal{U}(m; n) \models x^{d+1} = x^{im+d+1}$ , and therefore  $\mathcal{U}(m; n) \models xy^{d+1} = xy$ .  $\square$

In completing the proof of (t) we will use the following result (for example [5] or [9]).

**Lemma 1.6** If  $S$  is an additive groupoid of positive integers and  $d = \gcd(S)$ , then:

- (i)  $\gcd(N) = d$  for any generating subset  $N$  of  $S$ .
- (ii) There exists the least generating subset  $K = \{n_1, n_2, \dots, n_k\}$  of  $S$ , and  $K$  is finite.
- (iii) There exists  $s \in S$  such that for each positive integer  $j$ ,  $s + jd \in S$ .  $\square$

**Lemma 1.7** If  $d_1, d_2, \dots, d_k$  are divisors of  $m$ , and  $d = \gcd(d_1, d_2, \dots, d_k)$ , then

$$\mathcal{U}(m; d_1, d_2, \dots, d_k) = \mathcal{U}(m; d).$$

**Proof.** The inclusion  $\mathcal{U}(m; d) \subseteq \mathcal{U}(m; d_1, d_2, \dots, d_k)$  follows as in Lemma 1.5. For the converse inclusion, denote by  $S$  the additive groupoid of positive integers generated by  $\{d_1, d_2, \dots, d_k\}$ . By Lemma 1.6 (i) and (r) we have  $\gcd(S) = d$ , and  $\mathcal{U}(m; d_1, d_2, \dots, d_k) = \mathcal{U}(m; S)$ . By Lemma 1.6 (iii) there exists  $s \in S$  such that  $ms + d \in S$  and thus, by Lemma 1.1,  $\mathcal{U}(m; S) \models y^{ms+d+1} = y^{d+1}$ .  $\square$

Finally, by (l), (r), Lemma 1.5 and Lemma 1.7, it follows that

$$\mathcal{U}(M; N) = \mathcal{U}(m; n),$$

where  $m = \gcd(M)$  and  $n = \gcd(M \cup N)$ . This completes the proof of (t).

We note that the following equality holds in  $\mathcal{U}(m; m)$

$$(x^{m+1})^{m+1} = x^{m+1}, \quad (2)$$

(or more generally, in  $\mathcal{U}(m; n)$ , where  $n|m$ , the equality  $(x^{in+1})^{m+1} = x^{in+1}$  holds.)

The results obtained in Theorem 1 suggest saying that

$$x^{m+1}y = xy, \{xy^{n+1} = xy : n \in K\}, \{x^{m+1}y = xy, xy^{n+1} = xy\}$$

is the *canonical axiom system* of  $\mathcal{U}(M; \emptyset)$ ,  $\mathcal{U}(\emptyset; N)$ ,  $\mathcal{U}(M; N)$ , respectively, where  $M, N$  are nonempty sets of positive integers,  $m = \gcd(M)$ ,  $K$  is the least generating subset of  $\langle N \rangle$ , and  $n = \gcd(M \cup N)$ .

As a corollary of Theorem 1 (for example [2]) we obtain

**Corollary 1.1** *For any pair  $(M, N)$  the variety  $\mathcal{U}(M; N)$  is finitely based.  $\square$*

## 2 Closed Sets of Equations in $\mathcal{U}(M; N)$

The main result of this section is the following

**Theorem 2** *If  $M, N, M', N'$  are nonempty sets of positive integers, then:*

(i)  $\mathcal{U}(M; \emptyset) = \mathcal{U}(M'; \emptyset) \iff \gcd(M) = \gcd(M')$ .

(ii)  $\mathcal{U}(\emptyset; N) = \mathcal{U}(\emptyset; N') \iff \langle N \rangle = \langle N' \rangle$ .

(iii)  $\mathcal{U}(M; N) = \mathcal{U}(M'; N') \iff$

$$\gcd(M) = \gcd(M') \ \& \ \gcd(M \cup N) = \gcd(M' \cup N').$$

(iv)  $\mathcal{U}(M; \emptyset) \neq \mathcal{U}(\emptyset; N); \mathcal{U}(M; \emptyset) \neq \mathcal{U}(M'; N'); \mathcal{U}(\emptyset; N) \neq \mathcal{U}(M'; N')$ .

The  $\Leftarrow$ -parts of (i), (ii), (iii) hold by Theorem 1. The corresponding  $\Rightarrow$ -parts and (iv) are corollaries of the following statement, shown in [4] (Proposition 3.5).

**Proposition 2.1** *Let  $\mathbf{H}$  be a free groupoid in the variety  $\mathcal{U}(M; N)$ . Then the following statements hold:*

- (i) *If  $M \neq \emptyset$ ,  $N = \emptyset$ ,  $\gcd(M) = m$ , then a left equation  $x^{n+1}y = xy$  holds in  $\mathbf{H}$  iff  $m|n$ ; no right equation holds in  $\mathbf{H}$ .*

- (ii) If  $M = \emptyset$ ,  $N \neq \emptyset$ , then a right equation  $xy^{n+1} = xy$  holds in  $\mathbf{H}$  iff  $n \in \langle N \rangle$ ; no left equation holds in  $\mathbf{H}$ .
- (iii) If  $M \neq \emptyset$ ,  $N \neq \emptyset$  and  $m = \gcd(M)$ ,  $n = \gcd(M \cup N)$ , then  $x^{i+1}y = xy$  iff  $m|i$ , and  $xy^{j+1} = xy$  iff  $n|j$ , hold in  $\mathbf{H}$ .  $\square$

(We note that only-if parts of (i) and (iii) in Proposition 2.1 follow from the fact that  $C_n \in \mathcal{U}(n; \emptyset) \cap \mathcal{U}(kn; n)$ , where  $C_n$  is the groupoid that is the reduction of the cyclic group of order  $n$  to its binary operation.)

A set  $\Sigma$  of equations is said to be *closed* if, for every equation  $\varepsilon$ , the following implication holds:

$$(\Sigma \models \varepsilon) \Rightarrow (\varepsilon \in \Sigma).$$

**Proposition 2.2** (i) Assume that  $\Sigma$  is a set of equations containing at least one left equation and at least one right equation. Then  $\Sigma$  is a closed set iff there exist two positive integers  $m$  and  $n$  such that  $n$  is a divisor of  $m$  and

$$\Sigma = \{x^{im+1}y = xy : i \geq 1\} \cup \{xy^{jn+1} = xy : j \geq 1\}.$$

- (ii) A set  $\Sigma$  of left equations is closed iff there is a positive integer  $m$  such that

$$\Sigma = \{x^{im+1}y = xy : i \geq 1\}.$$

- (iii) A set  $\Sigma$  of right equations is closed iff there is an additive groupoid  $S$  of positive integers such that

$$\Sigma = \{xy^{n+1} = xy : n \in S\}. \quad \square$$

The lattices  $\mathcal{U}_l, \mathcal{U}_r, \mathcal{U}$  (of all left, right, two-sided varieties, respectively) can be characterized as follows:

**Proposition 2.3** (l)  $\mathcal{U}_l$  is isomorphic to the lattice of positive integers, where  $m \leq n$  iff  $m|n$ .

(r)  $\mathcal{U}_r$  is antiisomorphic to the lattice of additive groupoids of positive integers.

(t)  $\mathcal{U}$  is isomorphic to the lattice of pairs  $(m, n)$  of positive integers such that  $n$  is divisor of  $m$ , and:

$$(m, n) \leq (m', n') \iff m|m' \ \& \ n|n'. \quad \square$$

### 3 Incomplete $\mathcal{U}(M; N)$ –Groupoids and Varieties $\mathcal{U}(M; N)$ with Solvable Word Problem

We investigate here the class of incomplete  $\mathcal{U}(M; N)$ –groupoids and by applying the main result of Evans’s paper [6], we solve the word problem for some varieties  $\mathcal{U}(M; N)$ .

The term “*incomplete groupoid*” ([6]) has the same meaning as “*halfgroupoid*” ([1]) or “*partial groupoid*” ([8]). Namely, if  $G$  is a nonempty set,  $D$  a subset of  $G \times G$ , and  $\cdot : (x, y) \mapsto xy$  a map from  $D$  into  $G$ , then the pair  $\mathbf{G} = (G, \cdot)$  is called an *incomplete groupoid* with the *domain*  $D$ .

A groupoid  $\mathbf{H} = (H, \bullet)$  is called an *extension* of the incomplete groupoid  $\mathbf{G}$  iff  $G \subseteq H$  and  $a \bullet b = ab$ , for every  $(a, b) \in D$ . If  $G^o = G \cup \{0\}$ , where  $0 \notin G$ , then the groupoid  $\mathbf{G}^o = (G^o, \bullet)$  defined by

$$x \bullet y = \begin{cases} xy, & \text{if } (x, y) \in D \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

is an extension of  $\mathbf{G}$ . We call  $\mathbf{G}^o$  the *trivial extension* of  $\mathbf{G}$ .

If  $M, N$  are sets of positive integers such that  $M \cup N \neq \emptyset$ , then we denote by  $\mathcal{IU}(M; N)$  the *class of incomplete groupoids*  $\mathbf{G}$ , such that the corresponding trivial closure  $\mathbf{G}^o$  satisfies the following implications:

$$\begin{aligned} x^{m+1} \in G &\Rightarrow x^{m+1} \bullet y = x \bullet y, \\ y^{n+1} \in G &\Rightarrow x \bullet y^{n+1} = x \bullet y, \end{aligned} \quad (4)$$

for any  $m \in M, n \in N, x, y \in G$ .

Let  $\mathbf{G}$  be an incomplete groupoid and  $K$  a set of positive integers. We define an *equivalence*  $\sim_K$  on  $G$  as follows. If  $K = \emptyset$ , then  $\sim_K$  is the equality on  $G$ . If  $K \neq \emptyset$ , we define a relation  $\rightarrow_K$  on  $G$  by:

$$c \rightarrow_K d \iff d = c^{k+1}, \quad (5)$$

for  $c, d \in G$  and some  $k \in K$ , and we put:  $c \leftrightarrow_K d \iff (c \rightarrow_K d \text{ or } c \leftarrow_K d)$ . We denote by  $\sim_K$  the reflexive, symmetric and transitive closure of  $\rightarrow_K$  on  $G$ , i.e., the equivalence on  $G$  generated by  $\rightarrow_K$ .

By (3.1), (3.2), and (3.3), we obtain the following characterization of the class  $\mathcal{IU}(M; N)$ :

$$\mathbf{G} \in \mathcal{IU}(M; N) \iff (\forall x, x', y, y' \in G)(x \sim_M x' \ \& \ y \sim_N y' \Rightarrow xy = x'y') \quad (6)$$

Let  $\mathbf{G} \in \mathcal{IU}(M; N)$  and define

$$A = \{a \in G \mid a^{k+1} \in G, \text{ for every } k \in M \cup N\}, \quad B = G \setminus A; \quad (7)$$

clearly,  $B = \{b \in G \mid b^{k+1} \notin G, \text{ for some } k \in M \cup N\}$ .

By (3.1), (3.2) and (3.5) it follows that

$$\mathbf{G} \in \mathcal{IU}(M; N) \ \& \ A = G \Rightarrow \mathbf{G}^o \in \mathcal{U}(M; N). \quad (8)$$

Note that, in the special case when  $M = \{m\}$ ,  $N = \{n\}$ , and  $n|m$ , we have  $A = \{a \in G \mid a^{m+1} \in G\}$  and  $B = \{b \in G \mid b^{m+1} \notin G\}$ .

The following proposition is true.

**Proposition 3.1** (i) *If  $\mathbf{G} \in \mathcal{IU}(m; \emptyset)$ , then for each  $a \in A$ ,  $q \geq 0$ , and  $1 \leq k \leq m$ , the equality  $a^{q^{m+k+1}} = a^{k+1}$  holds.*

(ii) *If  $\mathbf{G} \in \mathcal{IU}(m; n)$ ,  $n|m$ , and  $a \in A$ , then  $(a^{in+1})^{m+1} = a^{in+1}$ .*

(iii)  $\mathcal{IU}(\emptyset; r, i) = \mathcal{IU}(\emptyset; r, i, r + i)$ .  $\square$

Using (3.3) and Proposition 3.1 we obtain the following

**Lemma 3.1** *Let  $\mathbf{G} \in \mathcal{IU}(m; n)$  and  $n|m$ . Then*

- (i)  $x \sim_m y \Rightarrow x^{m+1} = y^{m+1}$ ;
- (ii)  $x \sim_m y \Rightarrow x, y \in A \vee x = y \in B$ ,  
where  $\sim_m$  stands for  $\sim_{\{m\}}$ .

**Proof.** Let  $x \sim_m y$ . If  $x = y$ , then  $x^{m+1} = y^{m+1}$ . If  $x \neq y$ , then  $x \sim_m y \iff (\exists t_0, t_1, \dots, t_s \in G)x = t_0 \leftrightarrow t_1 \leftrightarrow \dots \leftrightarrow t_s = y$ , where  $\leftrightarrow$  stands for  $\leftrightarrow_{\{m\}}$ . The proof is given by induction on  $s$ . If  $s = 1$ , then  $x^{m+1} = y^{m+1}$ , and  $x, y \in A$ . If  $s = 2$ , we have the following four cases:

- 1)  $x \rightarrow t \rightarrow y$ ; then  $t = x^{m+1}$ ,  $y = t^{m+1}$ ,  $y = (x^{m+1})^{m+1} = x^{m+1}$   
(by Proposition 3.1), and thus  $y^{m+1} = (x^{m+1})^{m+1} = x^{m+1}$ ;
- 2)  $x \rightarrow t \leftarrow y$ ; then  $x^{m+1} = t = y^{m+1}$ ;
- 3)  $x \leftarrow t \leftarrow y$ ; then  $x^{m+1} = y^{m+1}$  follows by symmetry of 1);
- 4)  $x \leftarrow t \rightarrow y$ ; then  $x = t^{m+1} = y$ ;

and in each case  $x, y \in A$ .

If  $s > 2$ , then applying 1)–4), the sequence  $t_0, t_1, \dots, t_s$  can be reduced to a sequence with less than  $s + 1$  elements.  $\square$

As a corollary of Lemma 3.1 we obtain the following

**Proposition 3.2**

$$\mathbf{G} \in \mathcal{IU}(m; n) \ \& \ n|m \Rightarrow (\forall b, b' \in B)(b \sim_m b' \Rightarrow b = b'). \quad (9)$$

If  $b \in B$ , then we denote by  $p(b)$  the positive integer  $p$ , such that

$$b^p \neq 0, \ b^{p+1} = 0. \quad (10)$$

Now we are ready to prove the main result.

**Theorem 3** *If the pair  $(M, N)$  satisfies one of the following conditions*

- (i)  $M = \emptyset$ ,  $N \neq \emptyset$ ; (ii)  $M = \{m\}$ ,  $N = \emptyset$ ; (iii)  $M = \{m\} = N$ ,

*then for each (finite)  $\mathbf{G} \in \mathcal{IU}(M; N)$  there exists a (finite)  $\mathbf{H} \in \mathcal{U}(M; N)$  that is an extension of  $\mathbf{G}$ .*

**Proof.** If  $B = \emptyset$ , then  $\mathbf{G}^o$  is an extension of  $\mathbf{G}$ , finite if  $G$  is finite, such that, by (3.6),  $\mathbf{G}^o \in \mathcal{U}(M; N)$ . Thus, it remains to build an extension  $\mathbf{H} = (H, \bullet) \in \mathcal{U}(M; N)$ , assuming that  $B \neq \emptyset$ .

Consider first the case (i):  $M = \emptyset$ ,  $N \neq \emptyset$ .

Let  $L$  be a set such that  $L \cap G^o = \emptyset$ , and let  $b \mapsto \underline{b}$  be a surjection from  $B$  onto  $L$  with the following property:

$$(\forall b, c \in B)(\underline{b} = \underline{c} \iff b \sim c \ \& \ b^p = c^q), \quad (11)$$

where  $\sim$  is an abbreviation for  $\sim_N$ ,  $p = p(b)$ ,  $q = p(c)$ . Define an operation  $\bullet$  on  $H = G^o \cup L$  as follows:

- 1) If  $x, y \in G$ ,  $b \in B$ , then:
  - 1.1)  $x \bullet y = xy$ , for  $xy \in G$ ;
  - 1.2)  $x \bullet y = \underline{b}$ , for  $x = b^p, y \sim b$ .
- 2) If  $x \in G, b \in B$ , then:
  - 2.1)  $\underline{b} \bullet x = \underline{b}$ , for  $x \sim b$ ;
  - 2.2)  $x \bullet \underline{b} = x \bullet b$ , if  $x \bullet b$  is defined by 1.1) or 1.2).
- 3) If  $b, c \in B$ , and  $b \sim c$ , then  $\underline{b} \bullet \underline{c} = \underline{b}$ .
- 4)  $x \bullet y = 0$ , in any other case.

Using (3.9) and (3.4) one can directly show that  $\bullet$  is a well-defined operation on  $H$ .

It follows by 1.1) that  $\mathbf{H}$  is an extension of  $\mathbf{G}$ , and so it remains to show that  $\mathbf{H} \in \mathcal{U}(\emptyset; N)$ .

First, by (3.9) and the definition of  $\bullet$  we obtain the following properties:

- 5) If  $a \in A$ ,  $b \in B$ ,  $z \in L \cup \{0\}$ ,  $n \in N$ ,  $p = p(b)$ , then:
  - 5.1)  $a \bullet^{n+1} = a^{n+1}$ ;
  - 5.2)  $b \bullet^{n+1} = b^{n+1}$ , for  $n + 1 \leq p$ ;
  - 5.3)  $b \bullet^{n+1} = \underline{b}$ , for  $n + 1 > p$ ;
  - 5.4)  $z \bullet^k = z$ , for each  $k \in Z^+$ .

(Here,  $y_{\bullet}^k$  is the  $k$ -th power of  $y$  in  $\mathbf{H}$ , i.e.  $y_{\bullet}^1 = y$ ,  $y_{\bullet}^{k+1} = y_{\bullet}^k \bullet y$ .)

Now, by using properties 5) and the definition of  $\bullet$ , we can show that:

$$6) \ x \bullet (y_{\bullet}^{n+1}) = x \bullet y, \text{ for each } x, y \in H, n \in N, \text{ i.e. } \mathbf{H} \in \mathcal{U}(\emptyset; N).$$

Thus we have proved Theorem 3 in the case (i).

Now, consider the cases (ii)  $M = \{m\}$ ,  $N = \emptyset$  and (iii)  $M = N = \{m\}$ . The construction of a groupoid  $\mathbf{H} \in \mathcal{U}(M; N)$  that is an extension of  $\mathbf{G} \in \mathcal{UU}(M; N)$  is formally the same in case (ii) as in case (iii). In both cases we will denote the equivalence  $\sim_M$  in  $G$  by  $\sim$ ; and  $\approx$  is the equality in  $G$  in case (ii), and  $\approx$  is the same as  $\sim$  in case (iii).

Let

$$L = \{(b, i) : b \in B, p(b) < i \leq m\}$$

and  $H = G \cup L$ . (The union defining  $H$  is assumed to be disjoint.)

Define an operation  $\bullet$  in  $H$  as follows.

1') If  $x, y \in G$ , then:

$$1.1') \ x \bullet y = xy, \text{ if } xy \in G;$$

$$1.2') \ x \bullet y = b, \text{ if } b \in B, x \sim b^m, p(b) = m, y \approx b;$$

$$1.3') \ x \bullet y = (b, p(b) + 1), \text{ if } x \sim b^{p(b)}, p(b) < m, y \approx b.$$

2') If  $b \in B, y \in G, y \approx b$ , then:

$$2.1') \ (b, m) \bullet y = b;$$

$$2.2') \ (b, i) \bullet y = (b, i + 1), \text{ if } p(b) < i < m.$$

3') If  $x \in L$ , then  $x \bullet x = x$ .

4')  $x \bullet y = 0$ , in any other case.

Thus we obtain an extension  $\mathbf{H} = (H, \bullet)$  of  $\mathbf{G}$ . (The product  $x \bullet y$  for (ii) in the cases 1.2') and 1.3') is well-defined by (3.7).)

It remains to show that  $\mathbf{H} \in \mathcal{U}(M; N)$ .

For that purpose, note first that the following statements hold.

5') If  $a \in A, x \in B \cup L \cup \{0\}$ , then

$$5.1') \ a_{\bullet}^{m+1} = a^{m+1} \in G;$$

$$5.2') \ x_{\bullet}^{m+1} = x.$$

(Here, as in 5),  $y_{\bullet}^k$  is the  $k$ -th power of  $y$  in  $\mathbf{H}$ .)

We will now show that:

6')  $x \bullet^{m+1} \bullet y = x \bullet y$ , for any  $x, y \in H$ .

Namely, if  $x \in B \cup L \cup \{0\}$  or  $y \in L \cup \{0\}$ , then the equality 6') follows from 3'), 4') and 5.2'). There remains the case  $x \in A$ ,  $y \in G$ . Here, by 5.1') and the definition 1.1'), 1.2'), 1.3') and 4'), we obtain the desired equality 6').

Hence (in the case  $M = \{m\}$ ,  $N = \emptyset$ ),  $\mathbf{H} \in \mathcal{U}(m; \emptyset)$ .

It remains to show that, for  $M = N = \{m\}$ , the following identity holds in  $\mathbf{H}$ :

7')  $x \bullet (y \bullet^{m+1}) = x \bullet y$ .

By the same reasoning as for 6'), the equality 7') is true whenever  $y \in B \cup L \cup \{0\}$  or  $x = 0$ . For  $x \in G \cup L$  and  $y \in A$ , one can show that 7') is also true, in the same way as for 6').

Hence (in the case  $M = N = \{m\}$ ),  $\mathbf{H} \in \mathcal{U}(m; m)$ , and this completes the proof of Theorem 3.

The following statement is a special case of the main result of the paper [6]:

**Proposition 3.3** *If the pair  $(M, N)$  is such that for every  $\mathbf{G} \in \mathcal{IU}(M; N)$  there exists an extension  $\mathbf{H} \in \mathcal{U}(M; N)$ , then the word problem is solvable in the variety  $\mathcal{U}(M; N)$ .  $\square$*

As a corollary of Theorem 1, Proposition 3.3 and Theorem 3, we obtain the following

**Theorem 4** *If  $M \cup N$  is finite and one of the following conditions holds:*

(i)  $N = \emptyset$ ; (ii)  $M \neq \emptyset, N \neq \emptyset$ , and  $\gcd(M) = \gcd(M \cup N)$ ; (iii)  $M = \emptyset$ ,

*then the word problem is solvable in the variety  $\mathcal{U}(M; N)$ .  $\square$*

### Remarks

1. Theorem 1 and Theorem 2 suggest the following two questions:

a) Is the implication

$$\mathcal{U}(M; N) = \mathcal{U}(M'; N') \Rightarrow \mathcal{IU}(M; N) = \mathcal{IU}(M'; N')$$

true?

b) Is it true that, for every pair  $(M, N)$ , every  $\mathbf{G} \in \mathcal{IU}(M; N)$  has an extension  $\mathbf{H} \in \mathcal{U}(M; N)$ ?

The answer to both questions, in general, is negative, as the following example shows.

Let  $M$  be a nonempty set of positive integers,  $\gcd(M) = m$  and  $G = \{1, 2, \dots, m+1, m+2\}$ . Let  $\mathbf{G} = (G, \bullet)$  be an incomplete groupoid such that the corresponding canonical extension  $\mathbf{G}^\circ$  is defined as follows:

- $a_1)$   $i \bullet 1 = i + 1$ , if  $i = 1, 2, \dots, m + 1$ ;
- $a_2)$   $1 \bullet (m + 2) = 1$ ;
- $a_3)$   $(m + 1) \bullet (m + 2) = m + 1$ ;
- $a_4)$   $x \bullet y = 0$ , otherwise.

If  $m \notin M$  and  $p = \min(M) > m + 1$ , then  $x^{n+1} = 0$  for every  $x \in G$ ,  $n \in M$ , and thus, by (3.3),  $\mathbf{G} \in \mathcal{IU}(M; \emptyset)$ . On the other hand, we have  $1^{\bullet m+1} \bullet 1 = (m + 1) \bullet 1 = m + 2 \neq 2 = 1 \bullet 1$ , which implies that  $\mathbf{G} \notin \mathcal{IU}(m; \emptyset)$ . Hence,  $\mathcal{IU}(m; \emptyset) \not\subseteq \mathcal{IU}(M; \emptyset)$ , i.e. the answer to the question *a*) is negative.

Also,  $\mathbf{G} \in \mathcal{IU}(M; \emptyset)$  cannot be embedded in an  $\mathbf{H} \in \mathcal{U}(M; \emptyset)$  ( $= \mathcal{U}(m; \emptyset)$ ), because  $(1^{\bullet m+1}) \bullet 1 = m + 2 \neq 2 = 1 \bullet 1$ .

2. Theorem 3 and the main result of [7] imply that, for each of the cases: i)  $M \neq \emptyset$ ,  $N = \emptyset$ ; ii)  $M \neq \emptyset \neq N$ ,  $\gcd(M) = \gcd(M \cup N)$ ; iii)  $M = \emptyset$ ,  $N \neq \emptyset$ , the embeddability problem: “For a finite  $\mathbf{G} \in \mathcal{IU}(M; N)$ , is there an extension  $\mathbf{H} \in \mathcal{U}(M; N)$ ?” is solvable.
3. In connection with Theorem 4, the authors conjecture that, applying the main result of [7], one can obtain the following variant of **Theorem 4**: “If  $M \cup N$  is finite, then the word problem is solvable in  $\mathcal{U}(M; N)$ .”

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