Injective Groupoids in some Varieties of Groupoids

Ǵ. Čupona, N. Celakoski, B. Janeva

Abstract

Some varieties of groupoids are considered in this paper. In each variety $\mathcal{V}$ a class $\mathcal{V}$-inj is defined, such that the class of $\mathcal{V}$-free groupoids[1] is its proper subclass. For a groupoid $\mathbf{H} \in \mathcal{V}$ a set of $\mathcal{V}$-prime elements is also defined. Then, for each considered variety $\mathcal{V}$ a proposition, called *Bruck Theorem for the variety* $\mathcal{V}$, namely the following statement: *"A groupoid* $\mathbf{H} \in \mathcal{V}$ *is* $\mathcal{V}$*-free iff it satisfies the following two conditions:* (i) $\mathbf{H} \in \mathcal{V}$*-inj,* (ii) *The set of* $\mathcal{V}$*-prime elements in* H *generates* $\mathbf{H}$*."* is proved.

0 Introduction

Throughout this paper we assume that $\mathbf{F}$ is a given absolutely free groupoid with a basis B, i.e. groupoid free in the variety of all groupoids.

We are interested in a special case of Lemma 1.5 in [1], namely the following proposition.

Proposition 0.1 *A groupoid* $\mathbf{F}= (F, \cdot)$ *is absolutely free iff it satisfies the following conditions:*

(i) $\mathbf{F}$ *is injective*[2]

(ii) *The set* B *of prime elements*[3] *in* F *generates* $\mathbf{F}$.

Then B *is the unique (free) basis of* $\mathbf{F}$. $\square$

(We reffer to this proposition as Bruck's Theorem.)

A groupoid $\mathbf{G}= (G, *)$ is *associated* to $\mathbf{F}$ if it satisfies the following conditions:

---

[1] i.e. the class of groupoids free in the variety $\mathcal{V}$.

[2] A groupoid $\mathbf{G}=(G, \cdot)$ is *injective* iff the mapping $\cdot : (x, y) \mapsto x \cdot y$ is an injection.

[3] $a \in G$ is *prime* in $\mathbf{G}$ iff $(\forall x, y \in G) a \neq xy$.

(a) $B \subseteq G \subseteq F$,

(b) $(\forall t, u, \in F)\ (tu \in G \Rightarrow t, u \in G\ \&\ t * u = tu)$.

Let **G** be a groupoid associated to **F**. If **G** is $\mathcal{V}$-free with a basis B, then we say that **G** is a *$\mathcal{V}$-canonical groupoid.* (We note that there might exist more then one $\mathcal{V}$-canonical groupoids. However, as they are $\mathcal{V}$-free, they are isomorphic.) For a $\mathcal{V}$-canonical groupoid with a basis B we will use the notation $\mathbf{R}$=$(R, *)$, as R is obtained by a corresponding reduction on F, which depends, of course, on the variety $\mathcal{V}$.

For defining the class $\mathcal{V}$-inj of groupoids we essentially use properties of the corresponding $\mathcal{V}$-canonical groupoid **R**, formerly constructed. We will look for an axiom system of the class $\mathcal{V}$-inj among the properties of the $\mathcal{V}$-canonical groupoid **R** which are not related to the properties of $\mathcal{V}$-prime elements. If the identities that are the axioms of the variety $\mathcal{V}$ are normal[4], then $\mathcal{V}$-prime means the same as prime element according to Proposition 0.1.

Among the varieties considered in this paper, only the variety $\mathcal{U}_n$, $n \geq 2$ defined with the axiom[5] $x^n = x$ does not satisfy the above property, i.e. is not defined by a normal identity. If $\mathbf{G} \in \mathcal{U}_n$, we say that an element $a \in G$ is *$\mathcal{U}_n$-prime* iff

$$(\forall x, y \in G)\ (a = xy \Rightarrow x = y^{n-1}).$$

This paper consists of 3 sections.

In the first section we prove Bruck Theorem for the variety $\mathcal{U}_n$. At the same time we give a correction of the definition of the class $\mathcal{U}_n$-inj ($n \geq 3$) stated in [3], which does not provide validness of Bruck Theorem (see Example 1.1 below).

In the second section we define the class $\mathcal{V}_2$-inj for the variety $\mathcal{V}_2$ of groupoids defined by the axiom $(xy)^2 = x^2y^2$ and we prove the corresponding Bruck Theorem.

In section 3 we give a short overview of the results on injectivity in the varieties $\mathcal{U}$, $\mathcal{U}_r$ in [5] and [6]. This way we have in this paper all the results on injectivity of groupoids obtained up to now by the authors.

---

[4] An identity is said to be *normal* if neither of its sides is a variable.

[5] Throughout the paper x^n is defined by: $x^1 = x,\ x^{k+1} = x^k \cdot x$.

1 Injective Groupoids in $\mathcal{U}_n$

It is shown in [3] that the $\mathcal{U}_n$-canonical groupoid $\mathbf{R}= (R, *)$ is defined as follows:

$$R = \{t \in F \mid (\forall x \in F)\ x^n \notin P(t)\}.^6$$

If $t, u \in R$, then

$$t * u := \begin{cases} tu, & \text{if } t \neq u^{n-1} \\ u, & \text{if } t = u^{n-1} \end{cases}$$

determines an operation on R.

Note that for $t \in R$, t^k is the k-th power of t in $\mathbf{F}$. In the same way t_*^k is the k-th power of t in $\mathbf{R}$. Therefore: $t_*^1 = t$; $t_*^{k+1} = (t_*^k) * t$, and thus

$$(\forall t \in R,\ 1 \leq k < n)\ t_*^k = t^k, \tag{1}$$

which implies that for each $t \in R$,

$$|\{t, t_*^2, \ldots, t_*^{n-1}\}| = |\{t, t^2, \ldots, t^{n-1}\}| = n - 1. \tag{2}$$

Also,

$$(\forall v \in R \setminus B)(\exists!(t, u) \in R^2)\ v = t * u (= tu)\ \&\ t \neq u^{n-1}. \tag{3}$$

The properties (1.1), (1.2) and (1.3) suggest the following definition of the class $\mathcal{U}_n$-inj.

A groupoid $\mathbf{H}= (H, \cdot)$ is *injective* in $\mathcal{U}_n$ if it satisfies the following conditions:

i) $\mathbf{H} \in \mathcal{U}_n$;

ii) for each $a \in H$, the set $\{a, a^2, \ldots, a^{n-1}\}$ has exactly $n - 1$ elements;

iii) If $a \in H$ is not an $\mathcal{U}_n$-prime, then there is a uniquely determined pair $(b, c) \in H^2$, such that $a = bc$ & $b \neq c^{n-1}$.

(In this case we say that (b, c) is the *pair of divisors* of a, and write $(b, c)|a$.)

---

[6] For each $v \in F$, $P(v)$ and $|v|$ are defined as follows: $P(b) = \{b\}$, $P(tu) = \{tu\} \cup P(t) \cup P(u)$, and $|b| = 1$, $|tu| = |t| + |u|$, for each $b \in B$ and $t, u \in F$.

If an $\mathcal{U}_n$-injective groupoid is defined only by *i)* and *iii)*, as in [3], then the following example shows that Bruck Theorem can not be obtained.

Example 1.1 Let $n \geq 2,\ B = A \cup C,\ A \neq \emptyset,$

$$H := \{t \in F | (\forall a \in A, y \in F) a^2 \notin P(t)\ \&\ y^n \notin P(t)\}.$$

Define an operation $*$ in H by:

$$t * u := \begin{cases} tu, & \text{if } tu \in H \\ t, & \text{if } t = u \in A \\ u, & \text{if } t = u^{n-1} \end{cases}$$

Then **H**=$(H, *)$ satisfies *i)* and *iii)*, $B \neq \emptyset$ is the set of primes and generates **H**, but **H** is not $\mathcal{U}_n$–free.

Using (1.1), (1.2), (1.3) and the definition of the class $\mathcal{U}_n$-inj we obtain the following:

Proposition 1.2 *If* **H** *is* $\mathcal{U}_n$*-free, then* **H**$\in \mathcal{U}_n$*-inj.* □

Bellow we assume that **H**$\in \mathcal{U}_n$-inj.

Proposition 1.3 *If* (b, c) *is the pair of divisors of* $a \in H$*,* $a = c'd'\ \&\ (c', d') \neq (c, d)$*, then* $c' = d'^{n-1}$*.* □

Proposition 1.4 *For each* $a \in H,\ 2 \leq k \leq n-1,\ a^k$ *is not an* $\mathcal{U}_n$*-prime in* **H***, and* (a^{k-1}, a) *is the pair of divisors of* a^k *in* **H***.*

Proof. Let a^k be $\mathcal{U}_n$-prime. Then, as $a^k = a^{k-1} \cdot a$, we have $a^{k-1} = a^{n-1}$, which contradicts *ii)* of the definition of the class $\mathcal{U}_n$-inj. □

Now we assume that $\mathbf{H} \in \mathcal{U}_n$-inj is such that the set B of $\mathcal{U}_n$-primes in H is nonempty and generates **H**. If we put
$C_0 = B,\ C_1 = C_0C_0 = BB,$ and define C_{k+1} by

$$C_{k+1} = \{a \in H \setminus B :\ (c, d)|a \Rightarrow \{c, d\} \subseteq C_0 \cup C_1 \cup \cdots \cup C_k\ \&\ \{c, d\} \cap C_k \neq \emptyset),$$

then

$$H = \bigcup \{C_p \mid p \geq 0\}, \tag{4}$$

and $p \neq q \Rightarrow C_p \cap C_q = \emptyset$.

Also, by induction on i, it follows that

$$a \in C_k \Rightarrow (\forall i \leq n-1)\ a^i \in C_{k+i-1}, \tag{5}$$

which implies that $C_k \neq \emptyset$, for each $k \geq 0$.

Theorem 1 (Bruck Theorem for $\mathcal{U}_n$) *Let* $\mathbf{H} \in \mathcal{U}_n$. *Then* $\mathbf{H}$ *is* $\mathcal{U}_n$*-free iff* $\mathbf{H}$ *satisfies the following conditions*

(*i*) $\mathbf{H} \in \mathcal{U}_n$*-inj,*

(*ii*) *The set* B *of* $\mathcal{U}_n$*-primes in* H *is nonempty and generates* $\mathbf{H}$.

Proof. If $\mathbf{H}$ is $\mathcal{U}_n$-free, then by Proposition 1.1 we have that $\mathbf{H} \in \mathcal{U}_n$-inj, and the basis B of $\mathbf{H}$ is the set of $\mathcal{U}_n$-primes in $\mathbf{H}$ and generates $\mathbf{H}$.

Conversly, let $\mathbf{H} \in \mathcal{U}_n$-inj, and $B \neq \emptyset$ be the set of $\mathcal{U}_n$-primes in $\mathbf{H}$ and generates $\mathbf{H}$. Then, by (1.4), $H = \bigcup\{C_p \mid p \geq 0\}$.

Let $\mathbf{G} \in \mathcal{U}_n$ and $\lambda : B \to G$ be a mapping. For each $k \in N$ we define a sequence of mappings $\varphi_k : C_k \to G$ inductively as follows:

$\varphi_0 = \lambda$, and let φ_i be defined for each $i \leq k$.

If $a \in C_{k+1}$ and $(b, c)|a$ are such that $b \in C_r$ and $c \in C_s$, then $r, s \leq k$ and if we put $\varphi_{k+1}(a) = \varphi_r(b) \cdot \varphi_s(c)$, then $\varphi := \bigcup\{\varphi_i \mid i \geq 0\}$ is a mapping from H into G. If $a \in H$ is not a $\mathcal{U}_n$-prime and $(c, d)|a$, then $\varphi(a) = \varphi(c)\varphi(d)$.

Also, by induction on k, we have

$$\varphi(a^k) = (\varphi(a))^k, \tag{6}$$

for each $a \in H$ and $1 \leq k \leq n-1$.

It remains to prove that φ is a homomorphism. If $b, c \in H$, then either (b, c) is the pair of divisors of bc or $b = c^{n-1}$.

If (b, c) is the pair of divisors of bc, then $\varphi(bc) = \varphi(b)\varphi(c)$. On the other hand, if $b = c^{n-1}$, then

$\varphi(c^{n-1})\varphi(c) = \varphi(c)^{n-1}\varphi(c) = \varphi(c)^n = \varphi(c) = \varphi(c^n)$.

Thus in both cases possible we have

$$\varphi(bc) = \varphi(b)\varphi(c),$$

i.e. φ is a homomorphism from $\mathbf{H}$ into $\mathbf{G}$, and thus, $\mathbf{H}$ is $\mathcal{U}_n$-free with the basis B. □

We will give an example of an injective groupoid in $\mathcal{U}_n$ that is not $\mathcal{U}_n$-free.

Example 1.5 Let B be an infinite set and $\mathbf{R}= (R,*)$ the $\mathcal{U}_n$-canonical groupoid with the basis B. Define subsets $H \subseteq R$ and $D \subseteq H \times H$ as follows:

$$H := \{w \in R |\ |set(w)| = 1\}^7,$$

$$D := \{(x,y) \in H \times H \mid set(x) \neq set(y)\},$$

As $D \sim B$, there is an injection $\varphi : D \to B$. Using the operation $*$ in $\mathbf{R}$ and φ, we define an operation $\circ$ on H by:

$$x \circ y := \begin{cases} x * y, & \text{if } set(x) = set(y) \\ \varphi(x,y), & \text{if } set(x) \neq set(y), \end{cases}$$

and obtain that $(H, \circ) \in \mathcal{U}_n$-inj. If φ is a bijection, then the set of $\mathcal{U}_n$-primes is empty. Thus, by Theorem 1, $\mathbf{H}$ is not $\mathcal{U}_n$-free.

We note that if B is any set and $\varphi : D \to B$ a mapping, then: a) the groupoid $(H, \circ)$ constructed above belongs to $\mathcal{U}_n$; b) the set of $\mathcal{U}_n$-primes in $\mathbf{H}$ coincides with $B \setminus im\varphi$; c) $(H, \circ) \in \mathcal{U}_n$-inj iff φ is an injection. (In that case, since D is infinite, the set B must be infinite.)

Thus we have proved the following statement.

Corollary 1.6 *The class of $\mathcal{U}_n$-free groupoids is a proper subclass of the class $\mathcal{U}_n$-inj.* $\square$

2 Injective Groupoids in $\mathcal{V}_2$

We will give an axiom system for $\mathcal{V}_2$-inj, after introducing several notions.

If $\mathbf{G}= (G, \cdot)$ is a groupoid and $k \geq 0$, then $x \mapsto x^{(k)}$ is a transformation on G defined by:

$$x^{(0)} = x, \quad x^{(k+1)} = x^{(k)}x^{(k)} = (x^{(k)})^2. \tag{7}$$

---

[7] For each $w \in F$ we define $set(w)$ inductively as follows: $set(b) = \{b\}$, $set(uv) = set(u) \cup set(v)$, for each $b \in B,\ u, v \in F$

An element $b \in G$ is a *base* in **G** iff

$$(\forall x \in G)\ (b = x^{(p)} \Rightarrow p = 0). \tag{8}$$

If $a \in G$ and $a = b^{(k)}$, where b is a base, then we say that $k = [a]$ is an *exponent* of a, and $b = a^{(-k)}$ a *base of* a. (If **G**=**F**, then each element t has a unique base and a unique exponent.)

In [4] a construction of $\mathcal{V}_2$-canonical groupoid **R** with a basis B is given.

Namely, we define R as the least subset of R, such that $B \subseteq R$, and if $u = vw \in F \setminus B$, then:

$$u \in R \iff [v, w \in R \ \&\ (v = w \text{ or } min\{[v], [w]\} = 0)]. \tag{9}$$

We define an operation $*$ in R as follows:

If $u, v \in R,\ m = min\{[u], [v]\}$ then

$$u * v = (u^{(-m)}v^{(-m)})^{(m)}. \tag{10}$$

As a consequence of the properties of **R** and Theorem 2 in [4], an axiom system for the class $\mathcal{V}_2$-inj is obtained. Namely, we say that a groupoid **H** is *injective* in $\mathcal{V}_2$ iff it satisfies the following three conditions:

(0) $\mathbf{H} \in \mathcal{V}_2$,

(1) $(\forall a \in H)(\exists!(b,k) \in H \times N)$[8] $a = b^{(k)}$, where b is a base in **H**.
(In this case we say that $k = [a]$ is the *exponent* of a, and $b = a^{(-k)}$ *the base* of a.)

(2) If b is a base and b is not prime in **H**, then there is a unique pair $(c, d) \in H^2$, such that $b = cd$ and at least one among c and d is a base.
(In this case we say that (c, d) is the *pair of divisors* of the base b.)

We note that here, if x is a base, then $(x^{(p)}, x^{(p)})$ is the pair of divisors of $x^{(p+1)}$.

Considering the results in [4] and the definition of the class $\mathcal{V}_2$-inj, we have the following:

Proposition 2.1 *If* **H** *is* $\mathcal{V}_2$*-free with a basis* B*, then* **H**$\in \mathcal{V}_2$*-inj,* B *is the set of primes and generates* **H**. □

---

[8] N is the set of nonnegative integers.

Theorem 2 (Bruck Theorem for $\mathcal{V}_2$) *Let* $\mathbf{H} \in \mathcal{V}_2$*.* $\mathbf{H}$ *is* $\mathcal{V}_2$*-free iff it satisfies the following two conditions:*

(*i*) $\mathbf{H} \in \mathcal{V}_2$*-inj,*

(*ii*) *The set* B *of* $\mathcal{V}_2$*-primes in* $\mathbf{H}$ *is nonempty and generates* $\mathbf{H}$*.*

Proof. By Proposition 2.1 we have that each $\mathcal{V}_2$-free groupoid satisfies the two conditions.

To prove the converse, we construct a sequence of disjoint sets $(C_i|i \geq 0)$ as in the proof of Theorem 1. Then $H = \bigcup_{i\geq 0} C_i$, and

$$a \in C_k \Rightarrow (\forall p \in N)\ a^{(p)} \in C_{k+p}.$$

Now, if $\mathbf{G} \in \mathcal{V}_2$, we construct a sequence of mappings $\varphi_k : C_k \to G$. Then, as in Theorem 1., $\varphi = \bigcup\{\varphi_i \mid i \geq 0\}$ is the homomorphic extension of λ from $\mathbf{H}$ into $\mathbf{G}$, and thus, $\mathbf{H}$ is $\mathcal{V}_2$-free with the basis B. □

We give below an example of a $\mathcal{V}_2$-injective groupoid that is not $\mathcal{V}_2$-free.

Example 2.2 Recall ([4]) that each element u in $\mathbf{R}$ (the canonical $\mathcal{V}_2$-free groupoid with a basis B) has a unique base and a uniquely determined exponent, denoted by $[u]$. Let $\mathbf{R}$ be the canonical $\mathcal{V}_2$-free groupoid with an infinite basis B. Define subsets $H \subseteq R$, and $D \subseteq H \times H$ as follows:

$$H := \{x \in R \mid |set(x)| = 1\};$$
$$D := \{(u,v) \in H \times H \mid set(u) \neq set(v), min\{[u],[v]\} = 0\}.$$

Then $D \sim B$ and there is an injection $\varphi : D \to B$. Define an operation $\circ$ as follows:

$$u \circ v := \begin{cases} u * v, & \text{if } set(u) = set(v) \\ (\varphi(u^{(-m)}, v^{(-m)}))^{(m)}, & \text{if } set(u) \neq set(v), m = min\{[u],[v]\} \end{cases}$$

Then $(H,\circ)$ is $\mathcal{V}_2$-injective. If φ is a bijection, the set of primes is empty, and thus, by Bruck Theorem, $(H,\circ)$ is not $\mathcal{V}_2$-free.

Thus we have proved the following statement.

Corollary 2.3 *The class of* $\mathcal{V}_2$*-free groupoids is a proper subclass of the class* $\mathcal{V}_2$*-inj.* □

3 Injective Groupoids in $\mathcal{U}$ and $\mathcal{U}_r$

The varieties $\mathcal{U}_l$, $\mathcal{U}_r$ defined by $xy^2 = xy$, $x^2y = xy$ respectively are considered in [5] and $\mathcal{U}$ defined by $x^2y^2 = xy$ in [6][9]

Let $R = \{t \in F \mid (\forall \alpha, \beta \in F)\alpha\beta^2, \alpha^2\beta \notin P(t)\}$, and let an operation $*$ be defined in R by:

$$t * u = \begin{cases} tu, & \text{if } tu \in R \\ \alpha u, & \text{if } t = \alpha^2 \ \& \ \alpha u \in R \\ t\beta, & \text{if } u = \beta^2 \ \& \ t\beta \in R \\ \alpha\beta, & \text{if } t = \alpha^2 \ \& \ u = \beta^2 \ \& \ \alpha, \beta \in R. \end{cases}$$

Then $\mathbf{R} = (R, *)$ is the $\mathcal{U}$-canonical groupoid with the basis B (see 1.3, 1.4 in [6]).

This suggests the following definition of $\mathcal{U}$-injective groupoids.

A groupoid $\mathbf{H} \in \mathcal{U}$ is *$\mathcal{U}$-injective* iff for each element $a \in H$ which is not prime, there is a unique pair (b, c) of nonidempotent elements such that $a = bc$. In that case, $b = c$ iff a is an idempotent element.

(Then we say that (b, c) is the *pair of divisors of* a in $\mathbf{H}$ and we write $(b, c)|a$.)

The definition of $\mathcal{U}$-injective groupoids points out the following structural description of the $\mathcal{U}$-injective groupoids (Proposition 2.2 in [6]).

Proposition 3.1 *Let A and A' be two nonempty disjoint sets of the same cardinality, $\varphi : A \to A'$ a bijection, and $\psi : D \to A$ an injection, where*

$$D := \{(a, b) \mid a, b \in A, \ a \neq b\}.$$

If we define an operation $\bullet$ on the set $H = A \cup A'$ by:

$(\forall a, b \in A, a \neq b)$ $a \bullet b = \varphi(a) \bullet b = a \bullet \varphi(b) = \varphi(a) \bullet \varphi(b) = \psi(a, b)$,

$a \bullet a = \varphi(a)$,

then we obtain a $\mathcal{U}$-injective groupoid $\mathbf{H} = (H, \bullet)$ in which $A \setminus im\psi$ is the set of primes. (In this case we denote $\mathbf{H}$ by $(A, A'; \varphi, \psi)$.)

Conversely, if $\mathbf{H}$ is a $\mathcal{U}$-injective groupoid with at least two elements, then it is isomorphic with a groupoid $(A, A'; \varphi, \psi)$ defined as above. If ψ in $(A, A'; \varphi, \psi)$ is a bijection, then we obtain thet $(A, A'; \varphi, \psi)$ is a $\mathcal{U}$-injective groupoid which is not $\mathcal{U}$-free. □

---

[9] We note that $\mathcal{U} = \mathcal{U}_l \cap \mathcal{U}_r$.

We note that a $\mathcal{U}$-injective groupoid is finite with n elements iff $n = 1, 2, 4$ (see 2.3 in [6]).

In [5] a $\mathcal{U}_r$-canonical groupoid with a basis B is constructed and the identity $xy^k = xy$, for every $k \geq 1$ is proved. This enables us to state the following system of axioms for the class $\mathcal{U}_r$-inj.

A groupoid $\mathbf{H}$ belongs to $\mathcal{U}_r$-inj iff

(0) $\mathbf{H} \in \mathcal{U}_r$.

(1) If $a \in H,\ m, n \geq 1$ are such that $a^m = a^n$, then $m = n$.

(2) For each $a \in H$ which is not prime in $\mathbf{H}$, there is a unique pair $(c, d) \in H^2$ such that $a = bc$ and c is a base in $\mathbf{H}$ and
$[(\forall (b', c') \in H^2)\ a = b'c' \Rightarrow b = b'$ & c is the base of c'.]

Here, an element c of a groupoid $\mathbf{H} \in \mathcal{U}_r$ is a *base* in $\mathbf{H}$ iff

$$(\forall d \in H) c = d^k \Rightarrow k = 1.$$

(We note that the axiom system for $\mathcal{U}_r$-inj in [5] is more "economical" one, but the later is more "convinient for applications"; anyway, they are equivalent.)

Bruck Theorem (for $\mathcal{U}$ and $\mathcal{U}_r$) (proved in [5] and [6]) can be shown here in the same way as for $\mathcal{U}_n$ in section 1.

At the end, we will state some remarks.

Remark 1. The varieties $\mathcal{U}$, $\mathcal{U}_l$ and $\mathcal{U}_r$ are special cases of the variety $\mathcal{V}^{(m,n)}$ defined by $x^m y^n = xy$, where $m, n \geq 1$ [9].

Remark 2. The groupoids constructed in Example 1.5 and Example 2.2 depend essentially on the corresponding canonical groupoid. If we have constructed an example of $\mathcal{V}$-injective groupoid that is not $\mathcal{V}$-free not depending on the $\mathcal{V}$-canonical groupoid, we might be able to give a description of the class $\mathcal{V}$-inj (as for the class $\mathcal{U}$-inj in Proposition 3.1)..

Remark 3 The authors have investigated other varieties, as well (e.x. [7], [8] and [9]) and have given a description of $\mathcal{V}$-canonical groupoids.

Remark 4. In the varieties of left-zero groupoids (defined by $xy = x$) and constant groupoids (defined by $xy = uv$) each groupoid is free, and thus the class of $\mathcal{V}$-free groupoids coincides with the class of $\mathcal{V}$-inj. Bruck Theorem is obviously valid in these two cases, but no subclass different from the variety $\mathcal{V}$ could be obtained.

References

[1] R.H.Bruck: *A Survey of Binary Systems*, Springer-Verlag, 1958

[2] P.M.Cohn: *Universal Algebra*, Harpers Series in Modern Math., 1965

[3] Ǵ. Čupona, N. Celakoski: *Free Groupoids with* $x^n = x$, Proceedings of the I Congress of Mathematicians and Informaticians of Macedonia,(1996),5-16

[4] Ǵ. Čupona, N. Celakoski: *Free Groupoids with* $(xy)^2 = x^2y^2$, Contributions, Sec. Math. Tech. Sci., MANU, 17, 1-2(1996),5-17

[5] Ǵ. Čupona, N. Celakoski: *Free Groupoids with* $xy^2 = xy$, Bilten SDMI 21 (XXI) 1997, 5-16

[6] Ǵ. Čupona, N. Celakoski: *On Groupoids with the Identity* $x^2y^2 = xy$, Contributions, Sec. Math. Tech. Sci., MANU, XVIII, 1-2(1997),5-15

[7] Ǵ. Čupona, N. Celakoski, B. Janeva: *Free Groupoids with the Axioms of the Form* $x^{m+1}y = xy$ *and/or* $xy^{n+1} = xy$, N.Sad J. of Math. Vol 29 No 2. (1999)131-147, Proc. VIII Conf. "Algebra & Logic" (Novi Sad 1998)

[8] Ǵ. Čupona, N. Celakoski, B. Janeva: *Varieties of Groupoids with the Axioms of the Form* $x^{m+1}y = xy$ *and/or* $xy^{n+1} = xy$, Matematicki glasnik, (received by the editors)

[9] Ǵ. Čupona, N. Celakoski, B. Janeva: *Canonical Groupoids with* $x^my^n = xy$, Bull. Math. (1999)