FREE BIASSOCIATIVE GRUPOIDS*

Snežana Ilić [1], **Biljana Janeva** [2], **Naum Celakoski** [3]

[1] *Faculty of Sciences and Mathematics, University of Niš, Niš, Serbia and Montenegro*
[2] *Faculty of Natural Sciences and Mathematics, University of Skopje, Skopje, Republic of Macedonia*
[3] *Faculty of Mechanical Engineering, University of Skopje, Skopje, Republic of Macedonia*

Abstract

The subject of this paper is a study of the variety of groupoids that have the following property: each subgroupoid generated by two elements is a subsemigroup. A construction of free objects in this variety is given. Free objects in the variety of idempotent and commutative groupoids with the aforementioned property are also constructed.

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0 Preliminaries

The idea of considering biassociative groupoids came out from [3], where monoassociative groupoids (i.e. groupoids with the property that each subgroupoid generated by one element is a subsemigroup) are investigated. The goal of this paper is a description of free objects in the varieties of groupoids with the property that each subgroupoid generated by

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a two element set is a subsemigroup. In order to accomplish this, some definitions, notations and facts on free semigroups wil be given below.

Let A be a nonempty set. Then the set of all finite (nonempty) sequences $(a_1, a_2, \ldots, a_n)$, where $a_\nu \in A$, will be denoted by A^+. The pair $(A^+, \cdot)$, where "$\cdot$" is the concatenation of sequences, is a free semigroup with the basis A. In the sequel, A^+ will denote the semigroup and its carrier, as well, and the element $(a_1, a_2, \ldots, a_n)$ of A^+ will be denoted simply by $a_1a_2\ldots a_n$, or a^n in the case $a_1 = a_2 = \ldots = a_n = a$.

The following propositions are true.

Proposition 0.1 (a) *The semigroup A^+ is cancellative.*

(b) *For each $a \in A^+$ there is a unique pair $(b,k) \in A^+ \times \mathcal{N}$,*[1] *such that $a = b^k$, where $b \neq c^r$, for any $c \in A^+$ and $r \in \mathcal{N}\backslash\{1\}$.*

(c) *If $B \neq \emptyset$ and $B \subseteq C$, then $B^+ \subseteq C^+$.*

(d) $B \cap C \neq \emptyset \Rightarrow (B \cap C)^+ = B^+ \cap C^+$. $\square$

In the assertion (b), b is called the *base* and k the *exponent* of a. An element $u \in A^+$ is said to be *primitive in* A^+ if and only if $(\forall v \in A^+, n \geq 2)\,(u \neq v^n)$. The notion of primitive element could be introduced for any semigroup S just substituing A^+ by $\mathcal{S}$ in the definition above.

A groupoid $\mathbf{G} = (G, \cdot)$ is said to be *biassociative* if and only if (shorter iff) for any $a, b \in G$, the subgroupoid S of $\mathbf{G}$ generated by a and b, i.e. $S = \langle a, b\rangle$, is a subsemigroup of $\mathbf{G}$. Moreover, if S is commutative (idempotent, commutative and idempotent) subsemigroup of $\mathbf{G}$, then $\mathbf{G}$ is said to be *commutative* (*idempotent, commutative idempotent*) *biassociative groupoid*, respectively. The class of all biassociative (commutative, idempotent, commutative and idempotent) groupoids will be denoted by *Bass* (*ComBass*, *IdBass*, *ComIdBass*), respectively.

Let $\mathbf{G}= (G, \cdot) \in Bass$ and $a, b \in G$. The subsemigroup C of $\mathbf{G}$, generated by a, i.e. $C = \langle a\rangle$, is described by $C = \{a^k \,|\, k \geq 1\}$. The subsemigroup S of $\mathbf{G}$ generated by a, b, i.e. $S = \langle a, b\rangle$, in the case when $a \notin \langle b\rangle$ and $b \notin \langle a\rangle$ consists of all elements of the form $a^{\alpha_1}b^{\beta_1}\ldots a^{\alpha_r}b^{\beta_r}$, where $\alpha_1, \beta_r \geq 0$, $\beta_1, \alpha_2, \ldots, \beta_{r-1}, \alpha_r \geq 1$, and "$x^0$" means "lack of any symbol".

The class of biassociative groupoids is hereditary and closed under direct products and homomorphisms. Therefore:

[1] $\mathcal{N}$ is the set of positive integers.

Proposition 0.2 *The class of all biassociative groupoids is a variety.* □

The following proposition is also true.

Proposition 0.3 *If* $1 \leq |B| \leq 2$*, then* B^+ *is a free object in Bass with the basis* B*.* □

The corresponding proposition to 0.3 for *ComIdBass* is the following

Proposition 0.4 *If* $|B| = 1$*, then a free ComIdBass with the basis* B *is* B *itself. If* $B = \{a, b\}$*,* $a \neq b$*, then a free ComIdBass with the basis* B *is* $\{a, b, ab\}$*.* □

Considering Proposition 0.3 (Proposition 0.4), we will give in section 1 (section 2) the construction of a free groupoid in *Bass* (in *ComIdBass*) with a basis B, such that $|B| \geq 3$.

For this purpose we need some more definitions.

Let $G \neq \emptyset$, $D \subseteq G \times G$, and $\cdot : D \to G$ be a mapping. Then $\mathbf{G}= (G, D, \cdot)$ is called a *partial groupoid* with the *domain* D. A subset $P \subseteq G$ is said to be a *subgroupoid of the partial groupoid* $\mathbf{G}$ iff

$$(a, b) \in P^2 \cap D \Rightarrow a \cdot b \in P.$$

A subgroupoid of a partial groupoid need not be a groupoid, but it is a partial groupoid with the domain $P^2 \cap D$.

Let $\mathbf{S}= (S, D, \cdot)$ be a partial groupoid. $\mathbf{S}$ is called a *partial semigroup*[2] if and only if

$$(\forall a, b, c \in S)((ab)c, a(bc) \in S \Rightarrow (ab)c = a(bc)). \qquad (1)$$

Let P be a subgroupoid of a partial groupoid $\mathbf{G}$. If $\mathbf{P}$ is a partial semigroup, then $\mathbf{P}$ is called a *partial subsemigroup* of $\mathbf{G}$.

A partial groupoid $\mathbf{G}= (G, D, \cdot)$ is said to be a *partial commutative* (*idempotent, commutative idempotent*) *groupoid* iff

$$(\forall a, b \in G)(ab \in G \Rightarrow ba \in G \wedge ab = ba),$$

[2]A partial semigroup $\mathbf{S}= (S, D, \cdot)$ could be defined as follows

$$(\forall a, b, c \in S)((ab)c \in S \Rightarrow a(bc) \in S \wedge (ab)c = a(bc)),$$

but in this paper we will consider the one satisfying (1).

$$((\forall a \in G)(a^2 \in G \Rightarrow a = a^2),$$
$$(\forall a, b \in G)(ab, a^2 \in G \Rightarrow ba \in G \wedge ab = ba \wedge a^2 = a)),$$

respectively.

The following proposition is also true.

Proposition 0.5 *Let K, P be subgroupoids of the partial groupoid* $\mathbf{G}=(G, D, \cdot)$. *If $K \cap P \neq \emptyset$, then $K \cap P$ is a subgroupoid of* $\mathbf{G}$. $\square$

Let $\mathbf{G}$ be a partial groupoid, $\emptyset \neq A \subseteq G$, $\{P_i \mid i \in I\}$ the family of all subgroupoids of $\mathbf{G}$ containing A, and $P = \bigcap_{i \in I} P_i$. Then $P \neq \emptyset$, and (by Proposition 0.5) P is a subgroupoid of $\mathbf{G}$ which is called the *subgroupoid of* $\mathbf{G}$ *generated by* A and is denoted by $P = \langle A \rangle$.

If $\mathbf{G}=(G, D, \cdot)$ and $\mathbf{G}'=(G', D', \cdot)$ are partial groupoids and $\varphi : G \to G'$ is a mapping, then φ is called a *partial homomorphism* from $\mathbf{G}$ into $\mathbf{G}'$ iff

$$(\forall x, y \in G)((x, y) \in D, (\varphi(x), \varphi(y)) \in D' \Rightarrow \varphi(xy) = \varphi(x)\varphi(y)). \quad (2)$$

Using the notions of subgroupoid of a partial groupoid generated by a nonempty set and partial homomorphism, one can define a partial free object in a class of partial groupoids in a usual way.

In order to give constructions of free objhects in the varieties *Bass* and *ComIdBass* we need definitions of a partial biassociative groupoid and a free partial biassociative groupoid.

A partial groupoid $\mathbf{G}=(G, D, \cdot)$ is said to be *partial biassociative groupoid* (or partial *Bass*-groupoid) iff for any $a, b \in G$, $\langle a, b \rangle$ is a partial subsemigroup of $\mathbf{G}$.

A partial *Bass*-groupoid $\mathbf{H}$ is said to be a *free partial Bass-groupoid with the basis* B $(\neq \emptyset)$, if $\mathbf{H}$ is generated by B and if $\mathbf{G} \in Bass$ and $\lambda : B \to G$ is a mapping, then there is a (unique) mapping $\varphi : H \to G$, such that φ is a partial homomorphism that is an extension of λ.

1 Construction of a free biassociative groupoid

The construction of a free biassociative groupoid with a given basis B will be given only for $|B| \geq 3$, as it was mentioned in section 0. It will be given in several steps. In fact, an inductive construction of a chain

$H_0, H_1, \dots, H_k, \dots$ of partial biassociative groupoids will be given such that its union will be a free object in *Bass* with the basis B.

The first step will be the construction of H_1. To make the reading easier, we give the full construction when $|B| = 3,\ B = \{a, b, c\}$, and then we give just a short note for the case $|B| > 3$. Some auxiliary assertions in this section will be marked as 1.x.x.

1.1 Construction of H_1

The set $B = \{a, b, c\}$ has no structure, so it is asumed that $H_0 = B$ is a partial groupoid with the domain $D_0 = \emptyset$. Define the set H_1 by:

$$H_1 = \{a, b\}^+ \cup \{a, c\}^+ \cup \{b, c\}^+$$

(or, in general, $H_1 = \bigcup\{\{x, y\}^+ \mid x, y \in H_0,\ x \neq y\}$).

The fact that H_1 is a union of infinite sets, each being a free semigroup with a two element basis, implies that:

1.1.1 $\mathbf{H}_1 = (H_1, D_1, \cdot)$ *is a partial groupoid with the domain*

$$D_1 = \{(t, u) \mid \{t, u\} \subseteq \{a, b\}^+ \ \vee\ \{t, u\} \subseteq \{a, c\}^+ \ \vee\ \{t, u\} \subseteq \{b, c\}^+\},$$

(*or, in general,* $D_1 = \bigcup\{(\{x, y\}^+)^2 \mid x, y \in H_0,\ x \neq y\}$). □

Note that H_1 is a union (in general not disjoint) of free semigroups. It is not a groupoid, in the case $|B| \geq 3$. For example, if $a, b, c \in B,\ a \neq b \neq c \neq a$, then $ab, bc \in H_1$, but $(ab, bc) \notin D_1$, i.e. the "product" $ab \cdot bc$ does not exist in $\mathbf{H}_1$. The elements of B are primitive elements in $\mathbf{H}_1$, but there are other, such as $ab, bc, \dots$.

We give below some properties of $\mathbf{H}_1$.

1.1.2 $\mathbf{H}_1$ *is a partial Bass-groupoid and*

$$x, y \in H_1 \Rightarrow ((x, y) \in D_1 \iff (y, x) \in D_1).\ \square$$

The next proposition is true for H_1, but not for $H_k, k \geq 2$.

1.1.3 *If* $x, y, z \in H_1$*, then* $x(yz) \in H_1 \Rightarrow (xy)z \in H_1$*, and in this case,* $x(yz) = (xy)z$. □

1.1.4 $\mathbf{H}_1$ *is a free partial Bass-groupoid with the basis* B.

Proof. Clearly, B generates $\mathbf{H}_1$. Let $\mathbf{G} \in Bass$ and $\lambda : B \to G$ be a mapping. If $(x, y) \in D_1$, then $x, y \in \{u, v\}^+$, where $u, v \in B = \{a, b, c\}$. Since $\{u, v\}^+$ is afree semigroup with the basis $\{u, v\}$, there is a homomorphic extension ψ_1 of λ_1 from $\{u, v\}^+$ into $\mathbf{G}$, where λ_1 is the restriction of λ on the set $\{u, v\}$. We put $\varphi_1(xy) = \psi_1(xy) = \psi_1(x)\psi_1(y) = \varphi_1(x)\varphi_1(y)$. It is clear that φ_1 is a partial homomorphism from $\mathbf{H}_1$ into $\mathbf{G}$. $\square$

1.2 Construction of $\mathbf{H}_2$

Many "products" of elements of H_1 are not defined in H_1, such as $a \cdot (bc)$, $b \cdot (ac), (ab) \cdot (ac)$. To provide their existence, we extend H_1 to H_2 as follows:

$$H_2 = H_1 \cup (\cup\{\{t, u\}^+ \mid t, u \text{ are primitive elements in } H_1 \ \& \ (t, u) \notin D_1\}).$$

Remark 1. In the definition of H_2 we could have taken the union of the collection $\{\{v, w\}^+ \mid v, w \in H_1,\ (v, w) \notin D_1\}$, for if v, w are not primitive elements in H_1, then $v = t^m,\ w = u^n$ for some $t, u \in H_1$, and $\{v, w\}^+ \subseteq \{t, u\}^+$.

Remark 2. Denote $C_1 = \cup\{\{t, u\}^+ \mid t, u$ are primitive elements in H_1 & $(t, u) \notin D_1\}$. Then: $H_1 \cap C_1 = \{v^n \mid v$ is a primitive element in $H_1,\ n \geq 1\} \neq \emptyset$, $C_1 \setminus H_1$ is infinite. For example, the set

$$\cup\{t \cdot u \mid t, u \text{ are primitive elements in } H_1 \ \& \ (t, u) \notin D_1\}$$

is a proper subset of $C_1 \setminus H_1$.

Remark 3. If $v, w \in H_1$, then $v \cdot w$ is defined in H_2 iff $v \cdot w$ is defined in H_1 or $v \cdot w \in \{t, u\}^+$ for some primitive elements $t, u \in H_1$, such that $(t, u) \notin D_1$.

Remark 4. If t, u, v are primitive elements in H_1 such that $tu, uv \notin H_1$, then $(tu) \cdot v \notin H_2$ or $t \cdot (uv) \notin H_2$. Thus the domain D_2 of H_2 is:

$$D_2 = D_1 \cup (\cup\{(\{t, u\}^+)^2 \mid t, u \text{ are primitive elements in } H_1 \& (t, u) \notin D_1\}). \tag{3}$$

Note that the union in (3) need not be disjoint.

Some properties of H_2 will be listed bellow.

1.2.1 $\mathbf{H}_2$ *is a partial groupoid with the domain D_2, and $H_1^2 \subset D_2$.*[3] $\square$

[3] $A \subset B$ iff $A \subseteq B$ and $A \neq B$.

1.2.2 *Each element in H_2 has a uniquely determined base and exponent.* □

1.2.3 $\mathbf{H}_2$ *is a biassociative partial groupoid.* □

Note that $\mathbf{H}_2$ is not a partial semigroup, as $(ab)c \neq a(bc)$, although $(ab)c, a(bc) \in H_2$.

1.2.4 *If* $\mathbf{G} \in Bass$ *and* $\lambda : B \to G$ *is a mapping, then there is a unique partial homomorphism* $\varphi_2 : \mathbf{H}_2 \to \mathbf{G}$, *such that* φ_1 *is the restriction of* φ_2 *on the set* H_1 .

Proof. Let $\mathbf{G} \in Bass$, and $\lambda : B \to G$ be a mapping. Then $\varphi_1 : H_1 \to G$ is a partial homomorphism defined as in the proof of 1.1.4. If $x, y \in H_2$, $(x, y) \in D_2$ and $x, y \in \{u, v\}^+$, where u, v are primitive elements in H_1, then φ_2 is defined in the same way as φ_1 in 1.1.4. □

1.3 Construction of $H_n (n \geq 3)$

Assume that the partial *Bass* groupoids $B = H_0, H_1, \ldots, H_k$ are defined and the following conditions are satisfied:

a) For each $i, 0 \leq i \leq k-1,\ H_i^2 \subset D_{i+1}$.

b) For each $\mathbf{G} \in Bass$ *and* $\lambda : B \to G$, there is a chain of partial homomorphisms $\lambda = \varphi_0 \subseteq \varphi_1 \subseteq \ldots \subseteq \varphi_{k+1} \subseteq \ldots$, where $\varphi_k : H_k \to G$ for any $k \geq 0$.

Now, define H_{k+1} in the same way as H_2:

$$H_{k+1} = H_k \cup (\cup\{\{t, u\}^+ \mid t, u \text{ are primitive in } H_k \ \&\ (t, u) \notin D_k\}).$$

1.3.1 $\mathbf{H}_{k+1}$ *is a partial Bass-groupoid with the domain*

$$D_{k+1} = D_k \cup (\cup\{(\{t, u\}^+)^2 \mid t, u \textit{ are primitive in } H_k \ \&\ (t, u) \notin D_k\}).$$

□

Note that

$$D_{k+1} = H_k^2 \cup (\cup\{(\{t, u\}^+)^2 \mid t, u \text{ are primitive in } H_k \ \&\ (t, u) \notin D_k\}).$$

1.3.2 $(\forall k \geq 0)(H_k^2 \subset D_{k+1}$ *and* $D_k \subset H_k^2)$.

Proof. The proof will be given by induction on k for both statements at the same time.

Recall that $H_0 = B,\ D_0 = \emptyset$ and $D_1 = \cup\{(\{x,y\}^+)^2 \mid x,y \in H_0,\ x \neq y\}$. Clearly $D_0 \subset H_0^2$, and $((ab),b) \in D_1$, but $((ab),b) \notin H_0^2$, i.e. $H_0^2 \subset D_1$. Thus 1.3.2 is true for $k = 0$.

We also give the proof for $k = 1$, i.e. $H_1^2 \subset D_2$ and $D_1 \subset H_1^2$.

Since $H_1 = \{a,b\}^+ \cup \{a,c\}^+ \cup \{b,c\}^+$, it follows that $(ab,c) \in H_1^2$, but $(ab,c) \notin D_1$, and thus $D_1 \subset H_1^2$. It is easily seen that there are elements $x,y,u \in H_1 \setminus H_0$, such that $(x,y) \notin D_1$, and $u \in \{x,y\}^+$ (for example: $x = ab,\ y = ac,\ u = (ab)^2$ are in $H_1 \setminus H_0,\ (ab,ac) \notin D_1$ and $(ab)^2 \in \{ab,ac\}^+$). Then $(xy,u) \notin H_1^2$, but $(xy,u) \in D_2$, i.e 1.3.2 is true for $k = 1$.

Suppose that $H_r^2 \subset D_{r+1}$, and $D_r \subset H_r^2$, for each $r \in \{0,1,\ldots,k\}, k > 0$. We will prove that

$$H_{k+1}^2 \subset D_{k+2} \text{ and } D_{k+1} \subset H_{k+1}^2.$$

By the inductive hypothesis and the definitions of H_r, D_r, we have that $H_k \subset H_{k+1}$ and there are $x,y,u \in H_{k+1} \setminus H_k$, such that $(x,y) \notin D_k$ (as $D_k \subset H_k^2$) and $u \in \{x,y\}^+$. Then $(xy,u) \notin H_{k+1}^2$, but $(xy,u) \in D_{k+2}$. If $x,y,u \in H_{k+1} \setminus H_k$ are different primitive elements such that $u \notin \{x,y\}^+$, then $xy,u \in H_{k+1},\ (xy,u) \in H_{k+1}^2$, but $(xy,u) \notin D_{k+1}$. Thus, $D_{k+1} \subset H_{k+1}^2$. □

1.3.3 *Each element in H_{k+1} has a unique base and exponent.* □

1.3.4 *Let* $\mathbf{G} \in$ *Bass and* $\lambda : B \to G$ *be a mapping. Then there is a unique partial homomorphism* $\varphi_{k+1} : H_{k+1} \to G$*, such that* φ_k *is the restriction of* φ_{k+1} *on* H_k*, and* $\varphi_0 = \lambda$.

Proof. $\varphi_0 = \lambda$, and φ_1 has been defined in **1.1**. Let φ_k be defined and $(x,y) \in D_{k+1} \cap H_k^2$. Then $\varphi_{k+1}(xy) = \varphi_k(x)\varphi_k(y)$. If $(x,y) \in D_{k+1} \setminus H_k^2$, then $x,y \in \{u,v\}^+$, for some primitive elements $u,v \in H_k$, such that $(u,v) \notin D_k$. Thus, $xy = u^{\alpha_1}v^{\beta_1}\ldots u^{\alpha_r}v^{\beta_r}$, and we define

$$\varphi_{k+1}(xy) = \varphi_k(u)^{\alpha_1}\varphi_k(v)^{\beta_1}\ldots\varphi_k(v)^{\beta_r}.$$

It is clear that φ_{k+1} is a partial homomorphism, and φ_k is the restriction of φ_{k+1} on H_k. □

Theorem 1 *If $H = \bigcup_{k\geq 0} H_k$, then* **H** *is a free biassociative groupoid with the basis B.*

Proof. First, let $x, y \in H$. Then there is a $k \in \mathcal{N}$, such that $x, y \in H_k$ and by 1.3.2, $(x, y) \in D_{k+1}$. Thus $x \cdot y \in H_{k+1} \subseteq H$, i.e. **H** is a groupoid. Now, we will prove that $\mathbf{H} \in Bass$. Let $x, y \in H$, i.e. there is a k, such that$(x, y) \in D_k$. Then $\langle x, y\rangle$ is a subgroupoid of **H**. Let $u, v, w \in \langle x, y\rangle$. Then $(u, v), (uv, w), (v, w), (u, vw) \in D_s$, for some $s \geq k$. As H_k is a partial *Bass*-groupoid for each k, it follows that $(uv)w = u(vw) \in H_s \subseteq H$. Thus, $\langle x, y\rangle$ is a subsemigroup, i.e. $\mathbf{H} \in Bass$. Let $\mathbf{G} \in Bass$ and $\lambda : B \to G$ be a mapping. Define $\varphi : H \to G$ as follows. If $(x, y) \in D_k$, then $\varphi(xy) = \varphi_k(x)\varphi_k(y)$. It is clear that φ is a homomorphism, such that $\varphi_0 = \lambda$ is the restriction of φ on the set B . (Note that, by the construction, B generates **H**.) □

The following statements for $\mathbf{H}_k$ are also true.

1.3.5 *If $x, y \in H_k$, then $(x, y) \in D_k$ if and only if $(y, x) \in D_k$, and $\langle x, y\rangle$ is a subsemigroup of H_k.* □

1.3.6 $\mathbf{H}_k$ *is a cancellative partial groupoid, i.e.*

$$(x, y), (x, z) \in D_k \Rightarrow (xy = xz \Rightarrow y = z), \text{ and}$$

$$(x, z), (y, z) \in D_k \Rightarrow (xz = yz \Rightarrow x = y).$$

Proof. $\mathbf{H}_1$ is a cancellative groupoid. Let the statement be true for all $\mathbf{H}_r$, $r \leq k$, and let $(x, y), (x, z) \in D_{k+1} \setminus H_k^2$ and $xy = xz$. Then $x, y \in \{u, v\}^+$, for some primitive elements $u, v \in H_k$ such that $(u, v) \notin D_k$ and $xy = xz \in \{u, v\}^+$. As $\{u, v\}^+$ is a free semigroup generated by $\{u, v\}$, it is acanellative semigroup, and thus $y = z$. □

Remark 5. If we consider the class of *ComBass*, then Theorem 1 can be restated for *ComBass* by adding commutativity. The construction of a free commutative biassociative groupoid with a given basis B is essentially the same, except that it is based on a free commutative semigroup generated by two elements a and b, i.e. $\{a, b\}^{(+)}$ instead on a free semigroup $\{a, b\}^+$.

2 Construction of Free Commutative Idempotent Biassociative Groupoids

We will consider here the class of commutative idempotent biassociative groupoids (*ComIdBass*) defined in section 0. Clearly, if $\mathbf{G} \in ComIdBass$, then $\mathbf{G} \in Bass$ and $\mathbf{G}$ is commutative and idempotent groupoid. Considering Proposition 0.5, we obtain that:

$$\mathbf{G} \in ComIdBass \iff (\forall x, y \in G)\langle x, y\rangle = \{x, y, xy\},$$

where $xy = yx$.

Note that the following is validproposition holds.

Proposition 2.1 *If a, b are different objects, then the groupoid* $\mathbf{H}= (\{a, b, ab\}; \cdot)$ *defined by*

$\cdot$	a	b	ab
a	a	ab	ab
b	ab	b	ab
ab	ab	ab	ab

is a free semilattice with the basis $\{a, b\}$. □

We will consider the case $|B| = 3$. The case $|B| > 3$ will not be considered, as the construction of a free*ComIdBass*-groupoid with the basis B, is essentially the same as in the case $|B| = 3$.

Let $B = \{a, b, c\}$, $a \neq b \neq c \neq a$. We will construct a chain $H_0, H_1, \dots, H_k, \dots$ of partial *ComIdBass*-groupoids by induction on k.

Define $H_0 = B$ and a partial order $\leq_0$ by: $a <_0 b <_0 c$. H_0 is a partial *ComIdBass* groupoid with the domain $D_0 = \emptyset$. Put $H_1 = H_0 \cup \{ab, ac, bc\}$, and define $\leq_1$ to be the lexicographic order on H_1 generated by $\leq_0$. Then $\mathbf{H}_1 = (H_1, \cdot)$ is a partial *ComIdBass* groupoid with the domain

$$D_1 = \{(x, y) \mid x, y \in H_0\} = H_0^2.$$

Suppose that $\mathbf{H}_k$ and $\leq_k$ are defined such that $\mathbf{H}_k$ is a partial *ComIdBass*-groupoid. Define

$$H_{k+1} = H_k \cup \{x(yz) \mid x, yz \in H_k, x <_k yz, x \neq y, x \neq z, x \neq yz\} \quad (4)$$

and $\leq_{k+1}$ to be the lexicographic order on H_{k+1} generated by $\leq_k$.

Proposition 2.2 $\mathbf{H}_k$ *is a partial ComIdBass-groupoid, for any* $k \in \mathcal{N}$, *with the domain* $D_{k+1} = \{(x, y) \mid x, y \in H_k\} = H_k^2$.

Proof. $\mathbf{H}_0$ and H_1 are partial *ComIdBass* groupoids. Assume that $\mathbf{H}_k$ is a partial *ComIdBass* groupoid, and consider H_{k+1} defined by (4).

If $u, v \in H_{k+1},\ (u, v) \in D_{k+1}$, then $\{u, v, uv\} \subseteq H_{k+1}$. Thus $\mathbf{H}_{k+1}$ is a partial *ComIdBass*-groupoid. □

Proposition 2.3 (a) $H_k \subset H_{k+1}$, $\quad(b)$ $D_{k+1} \subset H_{k+1}^2$. □

Proposition 2.4 *If* $\mathbf{G} \in ComIdBass$ *and* $\lambda : B \to G$, *then for each* $k \geq 0$, *there is a partial homomorphism* $\varphi_{k+1} : H_{k+1} \to G$, *such that* φ_k *is the restriction of* φ_{k+1} *on* H_k *and* $\varphi_0 = \lambda$. □

Theorem 2 *Let* $H = \cup\{H_k \mid k \geq 0\}$. *Then* $\mathbf{H} = (H, \cdot)$ *is a free ComIdBass-groupoid with the basis* B.

Proof. In the same way as in Theorem 1, one can prove that $\mathbf{H} \in ComIdBass$, it is generated by B and if $\mathbf{G} \in ComIdBass$ and $\lambda : B \to G$ is a mapping, then $\varphi = \cup_{k \geq 0} \varphi_k : H \to G$ is the homomorphic extension of λ. □

Remark 6. For the construction of a free object in the variety *IdBass* with a basis B, a theorem similar to Theorem 2 can be used. Then the construction is essentially the same as for *ComIdBass*, except that here the free idempotent semigroup $\{a, b, ab, ba, aba, bab\}$ generated by $\{a, b\}$ is used, instead of a free commutative idempotent semigroup $\{a, b, ab\}$ generated by $\{a, b\}$.

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